1.12 Core Pure: Complex Numbers Depth: 1-hour Pack

Total Marks

EDEXCEL 9FM0 · Calculator allowed · about 60 minutes

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 Let $z = \sqrt{2}*(\cos(\pi/6) + i*\sin(\pi/6))$.

(a) Use de Moivre's theorem to find z^6, giving your answer in the form a + b*i, where a and b are integers. **(4)**

(b) Find the smallest positive integer n for which z^n is a positive real number. **(3)**

(Total for Question 1 is 7 marks)

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2 The complex number z satisfies $|z - (-1 + i)| = 5$.

(a) Describe the locus of z geometrically, and write its equation in Cartesian form. **(4)**

(b) Find the greatest value of |z| and the least value of |z|. **(3)**

(Total for Question 2 is 7 marks)

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3 The complex number z satisfies simultaneously $\arg(z - (1 + i)) = \pi/3$ and $|z - 6| = |z - 2*i|$.

(a) Show that the locus $|z - 6| = |z - 2*i|$ can be written as $y = 3*x - 8$, where $z = x + i*y$. **(3)**

(b) Show that the locus $\arg(z - (1 + i)) = \pi/3$ can be written as $y = \sqrt{3}*x + 1 - \sqrt{3}$, stating the restriction on x. **(3)**

(c) Hence find, in the form p + q*i where p and q are surds, the single complex number z satisfying both loci simultaneously. **(4)**

(Total for Question 3 is 10 marks)

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4 A transformation from the z-plane to the w-plane is defined by $w = 1/z$, where z is not equal to 0. The point z lies on the circle $|z - 3| = 3$ (excluding the origin).

(a) Show that, in the w-plane, the image of this circle is the line with equation $6*\text{Re}(w) = 1$. **(7)**

(b) State, with a reason, whether every point on the line $6*\text{Re}(w) = 1$ is the image of a point on the circle $|z-3|=3$ (excluding the origin). **(2)**

(c) Find the point z on the circle such that the image $w = 1/z$ also satisfies $\text{Im}(w) = 1/6$. **(4)**

(Total for Question 4 is 13 marks)

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5 This question uses de Moivre's theorem to derive and apply a multiple-angle identity in $\cos(\theta)$.

(a) Using de Moivre's theorem and the binomial expansion of $(\cos(\theta) + i*\sin(\theta))^5$, show that $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$. **(7)**

(b) Hence solve, for $0 \leq \theta < \pi$, the equation $16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta) = 0$, giving each solution as an exact multiple of π. **(5)**

(Total for Question 5 is 12 marks)

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Mark scheme · 1.12 Core Pure: Complex Numbers Depth: 1-hour Pack

Question 1

(a) $z^6 = (\sqrt{2})^6*(\cos(6*\pi/6) + i*\sin(6*\pi/6))$ **M1**

(a) $(\sqrt{2})^6 = 8$ **A1**

(a) simplifies the angle $6*\pi/6 = \pi$, giving $\cos(\pi) = -1$ and $\sin(\pi) = 0$ **M1**

(a) $z^6 = -8$, cao **A1**

(a) Answer: $z^6 = -8$

(b) $z^n = (\sqrt{2})^n*(\cos(n*\pi/6) + i*\sin(n*\pi/6))$, and requires $n*\pi/6$ to be a multiple of $2*\pi$ **M1**

(b) $n/6 = 2k$ for integer k, i.e. $n = 12k$ **M1**

(b) smallest positive integer n = 12, cao **A1**

(b) Answer: n = 12

Question 2

(a) a circle with centre (-1, 1) **B1**

(a) radius 5 **B1**

(a) writes $z = x + iy$ and uses $|z - w|^2$ **M1**

(a) $(x + 1)^2 + (y - 1)^2 = 25$ **A1**

(a) Answer: $(x + 1)^2 + (y - 1)^2 = 25$

(b) distance of the centre from the origin = $\sqrt{2}$ **M1**

(b) greatest = $\sqrt{2} + 5$ **A1**

(b) least = $|\sqrt{2} - 5|$ (the origin lies inside the circle when $\sqrt{2} < 5$) **A1**

(b) Answer: greatest $\sqrt{2} + 5 = 6.414$, least 3.586

Question 3

(a) $(x-6)^2 + y^2 = x^2 + (y-2)^2$ **M1**

(a) expands and simplifies to $-12x + 36 = -4y + 4$ **M1**

(a) $y = 3x - 8$, cso **A1**

(a) Answer: y = 3x - 8

(b) $\tan(\pi/3) = (y-1)/(x-1)$, with $x > 1$ (and $y > 1$) since arg = $\pi/3$ requires both components positive **M1**

(b) $y - 1 = \sqrt{3}*(x - 1)$ **A1**

(b) $y = \sqrt{3}*x + 1 - \sqrt{3}$, for $x > 1$, cso **A1**

(b) Answer: $y = \sqrt{3}*x + 1 - \sqrt{3}$, for $x > 1$

(c) sets $3x - 8 = \sqrt{3}*x + 1 - \sqrt{3}$ **M1**

(c) solves for x: $x*(3 - \sqrt{3}) = 9 - \sqrt{3}$ **M1**

(c) rationalises to $x = 4 + \sqrt{3}$ **A1**

(c) $y = 3x - 8 = 4 + 3*\sqrt{3}$, giving $z = (4 + \sqrt{3}) + i*(4 + 3*\sqrt{3})$, cao **A1**

(c) Answer: $z = (4 + \sqrt{3}) + i*(4 + 3*\sqrt{3})$

Question 4

(a) writes $z = x + i*y$ and expands $|z-3|=3$ as $(x-3)^2+y^2=9$, giving the Cartesian equation $x^2+y^2 = 6*x$ **M1**

(a) correct Cartesian equation $x^2+y^2 = 6*x$ **A1**

(a) writes $w = 1/z = (x-i*y)/(x^2+y^2)$, rationalising by the conjugate **M1**

(a) substitutes $x^2+y^2=6*x$ to get $w = (x-i*y)/(6*x)$ **A1**

(a) takes the real part: $\text{Re}(w) = x/(6*x) = 1/6$, valid whenever x is not 0 **M1**

(a) notes that x=0 on the circle occurs only at the origin, which is excluded, so the result holds for every image point **B1**

(a) $6*\text{Re}(w) = 1$, cso **A1**

(a) Answer: 6*Re(w) = 1, i.e. Re(w) = 1/6

(b) correct statement: yes, every point of the line is the image of exactly one point on the circle **B1**

(b) valid supporting reason, e.g. that w=1/z is a one-to-one map on the punctured plane (z not equal to 0), and it is a standard result that a circle through the origin (with the origin removed) maps onto the whole of a line under w=1/z **B1**

(b) Answer: Yes: every point on the line 6*Re(w) = 1 is the image of exactly one point on the circle (other than the origin itself, which has no finite image)

(c) combines with part (a): $w = 1/6 + i/6 = (1+i)/6$ **M1**

(c) $z = 1/w = 6/(1+i)$ **M1**

(c) rationalises: $z = 6*(1-i)/((1+i)*(1-i)) = 6*(1-i)/2 = 3*(1-i)$ **A1**

(c) $z = 3 - 3*i$, cao **A1**

(c) Answer: z = 3 - 3*i

Question 5

(a) writes $\cos(5*\theta) + i*\sin(5*\theta) = (\cos(\theta) + i*\sin(\theta))^5$ by de Moivre's theorem **M1**

(a) attempts the binomial expansion of the right-hand side **M1**

(a) correct unsimplified expansion: $\cos^5(\theta) + 5i*\cos^4(\theta)\sin(\theta) - 10\cos^3(\theta)\sin^2(\theta) - 10i*\cos^2(\theta)\sin^3(\theta) + 5\cos(\theta)\sin^4(\theta) + i*\sin^5(\theta)$ **A1**

(a) equates real parts: $\cos(5*\theta) = \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)$ **M1**

(a) substitutes $\sin^2(\theta) = 1 - \cos^2(\theta)$ throughout (dependent on the previous method mark) **dM1**

(a) correctly expands and collects terms in $\cos(\theta)$ **A1**

(a) $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$, cso **A1**

(a) Answer: $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$

(b) uses part (a) to rewrite the equation as $\cos(5*\theta) = 0$ **M1**

(b) sets the general solution $5*\theta = \pi/2 + k*\pi$ **M1**

(b) $\theta = \pi/10 + k*\pi/5$ **M1**

(b) at least three correct values within the range **A1**

(b) all five values correct with no extras: $\theta = \pi/10, 3*\pi/10, \pi/2, 7*\pi/10, 9*\pi/10$ **A1**

(b) Answer: $\theta = \pi/10, 3*\pi/10, \pi/2, 7*\pi/10, 9*\pi/10$

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