1.12 Core Pure: Complex Numbers Depth

EDEXCEL 9FM0 · Calculator allowed · about 230 minutes · Write-in edition

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 Let $z = 1 + i$.

(a) Find the modulus and argument of z. **(2)**

(b) Hence use de Moivre's theorem to find z^8, giving your answer as a real integer. You must show your method. **(2)**

(Total for Question 1 is 4 marks)

2 Let $z = \sqrt{2}*(\cos(\pi/6) + i*\sin(\pi/6))$.

(a) Use de Moivre's theorem to find z^6, giving your answer in the form $a + b*i$, where a and b are integers. **(4)**

(b) Find the smallest positive integer n for which z^n is a positive real number. **(3)**

(Total for Question 2 is 7 marks)

3 Let $w = -8 - 8*\sqrt{3}*i$.

(a) Express w in the form $r*(\cos(\theta) + i*\sin(\theta))$, where $r > 0$ and $-\pi < \theta \leq \pi$. **(3)**

(b) Hence use de Moivre's theorem to find, in the form $x + i*y$, the four roots of the equation $z^4 = w$. **(6)**

(c) Show that the sum of these four roots is 0, and explain briefly why this must be the case. **(3)**

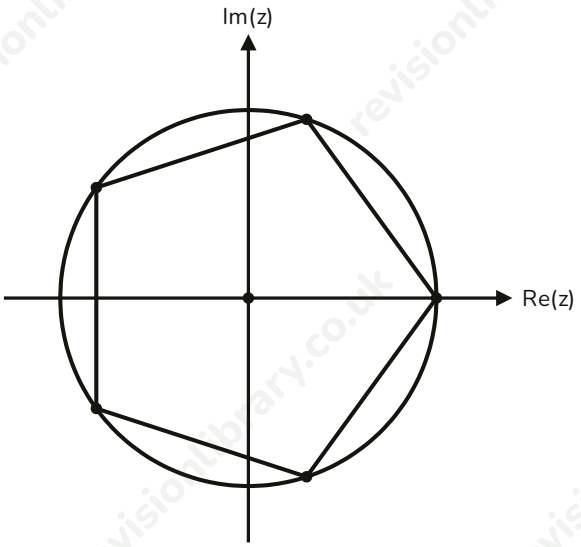
(Total for Question 3 is 12 marks)

4 The complex number $u = \cos(2*\pi/5) + i*\sin(2*\pi/5)$ is a fifth root of unity.

(a) Write down, in a similar form, the other four fifth roots of unity (i.e. the roots of $z^5 = 1$ other than u), for k = 0, 2, 3, 4. **(2)**

(b) Show that $1 + u + u^2 + u^3 + u^4 = 0$. **(3)**

(c) The five roots of $z^5 = 1$ are represented on an Argand diagram by points forming a regular pentagon inscribed in a circle of radius 1 centred at the origin. Show that the area of this pentagon is (5/2)*sin(2*π/5). **(5)**



(d) Hence find the area of the pentagon, giving your answer to 3 significant figures. **(2)**

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(Total for Question 4 is 12 marks)

5 The complex number z satisfies $|z - (-1 - 4i)| = 5$.

(a) Describe the locus of z geometrically, and write its equation in Cartesian form. **(4)**

(b) Find the greatest value of $|z|$ and the least value of $|z|$. **(3)**

(Total for Question 5 is 7 marks)

6 Solve the equation $z^6 = 1$, giving each root in the form $a + bi$ with a and b exact.

(a) Write 1 in modulus-argument form. **(2)**

(b) Hence find all 6 roots of the equation. **(4)**

(Total for Question 6 is 6 marks)

7 The complex numbers z_1 and z_2 are $z_1 = 1 + 6i$ and $z_2 = 4 + 3i$.

(a) Find the modulus and the argument of z_1, giving the argument in radians to 3 significant figures, in the range $-\pi < \arg \leq \pi$. **(4)**

(b) Find the modulus and argument of z_2 to 3 significant figures. **(3)**

(c) Hence find the modulus and argument of $z_1 z_2$ and of z_1/z_2, to 3 significant figures, with arguments in the range $-\pi < \arg \leq \pi$. **(3)**

(d) Find the modulus and argument of $(z_1)^4$ to 3 significant figures, with the argument in the range $-\pi < \arg \leq \pi$. **(3)**

(Total for Question 7 is 13 marks)

8 The complex number z satisfies $|z - (-1 + i)| = 5$.

(a) Describe the locus of z geometrically, and write its equation in Cartesian form. **(4)**

(b) Find the greatest value of $|z|$ and the least value of $|z|$. **(3)**

(Total for Question 8 is 7 marks)

9 The complex number z is $z = -\sqrt{3} + i$.

(a) Write z in modulus-argument form $r(\cos\theta + i\sin\theta)$, with θ in degrees. **(3)**

(b) Use de Moivre's theorem to find z^3, giving your answer in the form $a + bi$, with a and b exact. **(4)**

(Total for Question 9 is 7 marks)

10 The cubic equation $z^3 + pz^2 + qz - 26 = 0$, where p and q are real constants, has a root $-3 + 2i$.

(a) Write down another complex root of the equation, with a reason. **(2)**

(b) Find the third root of the equation. **(3)**

(c) Find the values of p and q. **(4)**

(Total for Question 10 is 9 marks)

11 The complex numbers z_1 and z_2 are given by $z_1 = 1 + 2i$ and $z_2 = 6 + 2i$.

(a) Find $z_1 z_2$ in the form $a + bi$. **(2)**

(b) Find z_1/z_2 in the form $a + bi$, where a and b are rational. **(3)**

(c) Find the two square roots of $-8 - 6i$, giving each in the form $a + bi$. **(4)**

(Total for Question 11 is 9 marks)

12 The complex numbers z_1 and z_2 are given by $z_1 = -1 - 3i$ and $z_2 = 2 + 2i$.

(a) Find $z_1 z_2$ in the form $a + bi$. **(2)**

(b) Find z_1/z_2 in the form $a + bi$, where a and b are rational. **(3)**

(c) Find the two square roots of $-3 - 4i$, giving each in the form $a + bi$. **(4)**

(Total for Question 12 is 9 marks)

13 The complex numbers z_1 and z_2 are $z_1 = -6 + i$ and $z_2 = 2 + 6i$.

(a) Find the modulus and the argument of z_1, giving the argument in radians to 3 significant figures, in the range $-\pi < \arg \leq \pi$. **(4)**

(b) Find the modulus and argument of z_2 to 3 significant figures. **(3)**

(c) Hence find the modulus and argument of $z_1 z_2$ and of z_1/z_2, to 3 significant figures, with arguments in the range $-\pi < \arg \leq \pi$. **(3)**

(d) Find the modulus and argument of $(z_1)^3$ to 3 significant figures, with the argument in the range $-\pi < \arg \leq \pi$. **(3)**

(Total for Question 13 is 13 marks)

14 Let $z1 = -1 + i*\sqrt{3}$ and $z2 = 2 - 2*i$.

(a) Express z1 in the form $r*(\cos(\theta) + i*\sin(\theta))$, where $r > 0$ and $-\pi < \theta \leq \pi$. **(3)**

(b) Express z2 in the form $r*(\cos(\theta) + i*\sin(\theta))$, where $r > 0$ and $-\pi < \theta \leq \pi$. **(3)**

(c) Using your answers to parts (a) and (b), find z1*z2 and z1/z2, giving each in the form $r*(\cos(\theta) + i*\sin(\theta))$ where $r > 0$ and $-\pi < \theta \leq \pi$. **(4)**

(d) Hence show that $z1/z2 = -(\sqrt{3}+1)/4 + i*(\sqrt{3}-1)/4$, using exact trigonometric values for $\cos(11*\pi/12)$ and $\sin(11*\pi/12)$. **(4)**

(Total for Question 14 is 14 marks)

15 The complex number z satisfies simultaneously $\arg(z - (1 + i)) = \pi/3$ and $|z - 6| = |z - 2*i|$.

(a) Show that the locus $|z - 6| = |z - 2*i|$ can be written as $y = 3*x - 8$, where $z = x + i*y$. **(3)**

(b) Show that the locus $\arg(z - (1 + i)) = \pi/3$ can be written as $y = \sqrt{3}*x + 1 - \sqrt{3}$, stating the restriction on x. **(3)**

(c) Hence find, in the form $p + q*i$ where p and q are surds, the single complex number z satisfying both loci simultaneously. **(4)**

(Total for Question 15 is 10 marks)

16 A transformation from the z-plane to the w-plane is defined by $w = 1/z$, where z is not equal to 0. The point z lies on the circle $|z - 3| = 3$ (excluding the origin).

(a) Show that, in the w-plane, the image of this circle is the line with equation $6*\mathrm{Re}(w) = 1$. **(7)**

(b) State, with a reason, whether every point on the line 6*Re(w) = 1 is the image of a point on the circle |z-3|=3 (excluding the origin). **(2)**

(c) Find the point z on the circle such that the image w = 1/z also satisfies Im(w) = 1/6. **(4)**

(Total for Question 16 is 13 marks)

17 This question uses de Moivre's theorem to derive and apply a multiple-angle identity in $\cos(\theta)$.

(a) Using de Moivre's theorem and the binomial expansion of $(\cos(\theta) + i*\sin(\theta))^5$, show that $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$. **(7)**

(b) Hence solve, for $0 \leq \theta < \pi$, the equation $16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta) = 0$, giving each solution as an exact multiple of π. **(5)**

(Total for Question 17 is 12 marks)

18 Let $z = \cos(\theta) + i*\sin(\theta)$.

(a) Show that $z - 1/z = 2*i*\sin(\theta)$. **(2)**

(b) Hence show that $32*i*\sin^5(\theta) = z^5 - 5*z^3 + 10*z - 10*z^{-1} + 5*z^{-3} - z^{-5}$. **(3)**

(c) Hence show that $\sin^5(\theta) = (1/16)*(\sin(5*\theta) - 5*\sin(3*\theta) + 10*\sin(\theta))$. **(4)**

(d) Hence find the exact value of the integral from $\theta = 0$ to $\theta = \pi/2$ of $\sin^5(\theta)$ with respect to θ. **(4)**

(Total for Question 18 is 13 marks)

Mark scheme · 1.12 Core Pure: Complex Numbers Depth

Question 1

(a) $|z| = \sqrt{1^2+1^2} = \sqrt{2}$ **B1**

(a) arg(z) = π/4, cao **B1**

(a) Answer: $|z| = \sqrt{2}$, arg(z) = π/4

(b) applies de Moivre's theorem: $z^8 = (\sqrt{2})^8*(\cos(8*\pi/4) + i*\sin(8*\pi/4))$, simplifying $(\sqrt{2})^8 = 16$ and $8*\pi/4 = 2*\pi$ **M1**

(b) $z^8 = 16$, cao **A1**

(b) Answer: $z^8 = 16$

Question 2

(a) $z^6 = (\sqrt{2})^6*(\cos(6*\pi/6) + i*\sin(6*\pi/6))$ **M1**

(a) $(\sqrt{2})^6 = 8$ **A1**

(a) simplifies the angle $6*\pi/6 = \pi$, giving $\cos(\pi) = -1$ and $\sin(\pi) = 0$ **M1**

(a) $z^6 = -8$, cao **A1**

(a) Answer: $z^6 = -8$

(b) $z^n = (\sqrt{2})^n*(\cos(n*\pi/6) + i*\sin(n*\pi/6))$, and requires $n*\pi/6$ to be a multiple of $2*\pi$ **M1**

(b) $n/6 = 2k$ for integer k, i.e. n = 12k **M1**

(b) smallest positive integer n = 12, cao **A1**

(b) Answer: n = 12

Question 3

(a) $|w| = \sqrt{8^2 + (8*\sqrt{3})^2} = \sqrt{64+192} = \sqrt{256} = 16$ **M1**

(a) reference angle $\arctan((8*\sqrt{3})/8) = \arctan(\sqrt{3}) = \pi/3$, third quadrant (both parts negative), so $\theta = -(\pi - \pi/3) = -2*\pi/3$ **A1**

(a) $w = 16*(\cos(-2*\pi/3) + i*\sin(-2*\pi/3))$, cao **A1**

(a) Answer: $w = 16*(\cos(-2*\pi/3) + i*\sin(-2*\pi/3))$

(b) uses $z_k = r^{1/4}*(\cos((\theta+2*k*\pi)/4) + i*\sin((\theta+2*k*\pi)/4))$ for k = 0, 1, 2, 3 **M1**

(b) finds $r^{1/4} = 16^{1/4} = 2$ and the angle formula $(-2*\pi/3 + 2*k*\pi)/4 = -\pi/6 + k*\pi/2$ **M1**

(b) any four correct arguments giving the four distinct Cartesian roots, e.g. $-\pi/6$, $\pi/3$, $5*\pi/6$ and $4*\pi/3$; equivalent arguments modulo $2*\pi$ accepted, with no principal-range requirement **A1**

(b) first two roots in x+iy form: $\sqrt{3}$ - i and 1 + $i*\sqrt{3}$ **A1**

(b) remaining two roots in x+iy form: $-\sqrt{3}$ + i and -1 - $i*\sqrt{3}$ **A1**

(b) all four roots correct and fully simplified, cao **A1**

(b) Answer: $z = \sqrt{3}$ - i, or $z = 1 + i*\sqrt{3}$, or $z = -\sqrt{3}$ + i, or $z = -1 - i*\sqrt{3}$

(c) adds real parts ($\sqrt{3}+1-\sqrt{3}-1 = 0$) and imaginary parts ($-1+\sqrt{3}+1-\sqrt{3} = 0$) **M1**

(c) sum = 0, cso **A1**

(c) explains that $z^4 - w = 0$ is a quartic in z with zero coefficient of z^3, and the sum of the roots of a quartic equals minus the coefficient of z^3 divided by the coefficient of z^4, which is 0 here **B1**

(c) Answer: Sum of the four roots = 0

Question 4

(a) roots given by $\cos(2*k*\pi/5) + i*\sin(2*k*\pi/5)$ for k = 0, 2, 3, 4, or equivalent forms **B1**

(a) correctly identifies the other four roots as 1, u^2, u^3, u^4 (or equivalent forms), excluding the given root u **B1**

(a) Answer: The other four roots are 1, u^2, u^3, u^4, i.e. $\cos(2*k*\pi/5) + i*\sin(2*k*\pi/5)$ for k = 0, 2, 3, 4

(b) recognises the sum as a geometric series with sum $(u^5 - 1)/(u - 1)$, valid since u is not equal to 1 **M1**

(b) $u^5 = 1$ (as u is a fifth root of unity), so the numerator is 0 **A1**

(b) hence the sum is 0, cso **A1**

(b) Answer: $1 + u + u^2 + u^3 + u^4 = 0$

(c) splits the pentagon into 5 congruent isosceles triangles from the centre, each with two sides of length 1 and included angle $2*\pi/5$ **M1**

(c) area of one triangle = $(1/2)*1*1*\sin(2*\pi/5)$, using Area = $(1/2)*a*b*\sin(C)$ **M1**

(c) correct triangle area $(1/2)*\sin(2*\pi/5)$ **A1**

(c) total pentagon area = $5*(1/2)*\sin(2*\pi/5)$ **M1**

(c) Area = $(5/2)*\sin(2*\pi/5)$, cso **A1**

(c) Answer: Area = $(5/2)*\sin(2*\pi/5)$

(d) evaluates $\sin(2*\pi/5)$, approximately 0.951056 **M1**

(d) area = awrt 2.38 (3 s.f.) **A1**

(d) Answer: Area is approximately 2.38 (3 s.f.)

Question 5

(a) a circle with centre (-1, -4) **B1**

(a) radius 5 **B1**

(a) writes z = x + iy and uses $|z - w|^2$ **M1**

(a) $(x + 1)^2 + (y + 4)^2 = 25$ **A1**

(a) Answer: $(x + 1)^2 + (y + 4)^2 = 25$

(b) distance of the centre from the origin = $\sqrt{17}$ **M1**

(b) greatest = $\sqrt{17} + 5$ **A1**

(b) least = $|\sqrt{17} - 5|$ (the origin lies inside the circle when $\sqrt{17} < 5$) **A1**

(b) Answer: greatest $\sqrt{17} + 5 = 9.123$, least 0.8769

Question 6

(a) modulus and argument **M1**

(a) 1(cos 0 + i sin 0) (plus multiples of 360 degrees) **A1**

(a) Answer: 1(cos 0 + i sin 0)

(b) modulus 1 and arguments (0 + 360k)/6 degrees for k = 0, 1, ..., 5 **M1**

(b) arguments 0, 60, 120, 180, 240, 300 **A1**

(b) uses exact values of cosine and sine **M1**

(b) roots: 1; 1/2 + (1/2)$\sqrt{3}$i; -1/2 + (1/2)$\sqrt{3}$i; -1; -1/2 - (1/2)$\sqrt{3}$i; 1/2 - (1/2)$\sqrt{3}$i **A1**

(b) Answer: 1; 1/2 + (1/2)$\sqrt{3}$i; -1/2 + (1/2)$\sqrt{3}$i; -1; -1/2 - (1/2)$\sqrt{3}$i; 1/2 - (1/2)$\sqrt{3}$i

Question 7

(a) modulus = $\sqrt{(1)^2 + (6)^2}$ **M1**

(a) $|z_1| = \sqrt{37} = 6.08$ **A1**

(a) uses the inverse tangent with a quadrant check **M1**

(a) arg $z_1 = 1.41$ **A1**

(a) Answer: $|z_1| = \sqrt{37}$, arg $z_1 = 1.41$

(b) modulus and argument by the same method **M1**

(b) $|z_2| = 5.00$ **A1**

(b) arg $z_2 = 0.644$ **A1**

(b) Answer: $|z_2| = 5.00$, arg $z_2 = 0.644$

(c) $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (adjusted to the range) **M1**

(c) $|z_1 z_2| = 30.4$, arg = 2.05 **A1**

(c) $|z_1/z_2| = 1.22$, arg = 0.762 **A1**

(c) Answer: $z_1 z_2$: modulus 30.4, arg 2.05; z_1/z_2: modulus 1.22, arg 0.762

(d) uses de Moivre: modulus to the power n, argument multiplied by n **M1**

(d) modulus 1370 **A1**

(d) arg = -0.661 **A1**

(d) Answer: modulus 1370, arg -0.661

Question 8

(a) a circle with centre (-1, 1) **B1**

(a) radius 5 **B1**

(a) writes z = x + iy and uses $|z - w|^2$ **M1**

(a) $(x + 1)^2 + (y - 1)^2 = 25$ **A1**

(a) Answer: $(x + 1)^2 + (y - 1)^2 = 25$

(b) distance of the centre from the origin = $\sqrt{2}$ **M1**

(b) greatest = $\sqrt{2} + 5$ **A1**

(b) least = $|\sqrt{2} - 5|$ (the origin lies inside the circle when $\sqrt{2} < 5$) **A1**

(b) Answer: greatest $\sqrt{2} + 5 = 6.414$, least 3.586

Question 9

(a) finds the modulus **M1**

(a) r = 2 **A1**

(a) θ = 150 degrees **A1**

(a) Answer: z = 2(cos 150 + i sin 150) degrees

(b) $z^3 = (2)^3$ (cos(3 x 150) + i sin(3 x 150)) **M1**

(b) modulus 8 and argument 450 degrees **A1**

(b) uses exact values of cos(90) and sin(90) **M1**

(b) 8i **A1**

(b) Answer: 8i

Question 10

(a) -3 - 2i **B1**

(a) complex roots of a polynomial with real coefficients occur in conjugate pairs **B1**

(a) Answer: -3 - 2i

(b) uses the product of the roots: (α)(α*)(r) = -(-26) **M1**

(b) $((-3)^2 + 2^2)$ r = 26 **A1**

(b) r = 2 **A1**

(b) Answer: 2

(c) uses sum of roots = -p **M1**

(c) p = 4 **A1**

(c) uses the sum of products of pairs = q **M1**

(c) q = 1 **A1**

(c) Answer: p = 4, q = 1

Question 11

(a) expands (a + bi)(c + di) with $i^2 = -1$ **M1**

(a) 2 + 14i **A1**

(a) Answer: 2 + 14i

(b) multiplies numerator and denominator by the conjugate of z_2 **M1**

(b) denominator 40 **A1**

(b) 1/4 + (1/4)i **A1**

(b) Answer: 1/4 + (1/4)i

(c) sets $(x + iy)^2 = -8 - 6i$, so $x^2 - y^2 = -8$ and $2xy = -6$ **M1**

(c) eliminates y to obtain a quadratic in x^2 (or finds a pair by inspection) **M1**

(c) x = -1, y = 3 **A1**

(c) -1 + 3i and 1 - 3i **A1**

(c) Answer: -1 + 3i and 1 - 3i

Question 12

(a) expands (a + bi)(c + di) with $i^2 = -1$ **M1**

(a) 4 - 8i **A1**

(a) Answer: 4 - 8i

(b) multiplies numerator and denominator by the conjugate of z_2 **M1**

(b) denominator 8 **A1**

(b) -1 - (1/2)i **A1**

(b) Answer: -1 - (1/2)i

(c) sets $(x + iy)^2 = -3 - 4i$, so $x^2 - y^2 = -3$ and $2xy = -4$ **M1**

(c) eliminates y to obtain a quadratic in x^2 (or finds a pair by inspection) **M1**

(c) x = -1, y = 2 **A1**

(c) -1 + 2i and 1 - 2i **A1**

(c) Answer: -1 + 2i and 1 - 2i

Question 13

(a) modulus = $\sqrt{(-6)^2 + (1)^2}$ **M1**

(a) $|z_1| = \sqrt{37} = 6.08$ **A1**

(a) uses the inverse tangent with a quadrant check **M1**

(a) arg $z_1 = 2.98$ **A1**

(a) Answer: $|z_1| = \sqrt{37}$, arg $z_1 = 2.98$

(b) modulus and argument by the same method **M1**

(b) $|z_2| = 6.32$ **A1**

(b) arg $z_2 = 1.25$ **A1**

(b) Answer: $|z_2| = 6.32$, arg $z_2 = 1.25$

(c) $|z_1 z_2| = |z_1||z_2|$ and $\arg(z_1 z_2) = \arg z_1 + \arg z_2$ (adjusted to the range) **M1**

(c) $|z_1 z_2| = 38.5$, arg = -2.06 **A1**

(c) $|z_1/z_2| = 0.962$, arg = 1.73 **A1**

(c) Answer: $z_1 z_2$: modulus 38.5, arg -2.06; z_1/z_2: modulus 0.962, arg 1.73

(d) uses de Moivre: modulus to the power n, argument multiplied by n **M1**

(d) modulus 225 **A1**

(d) arg = 2.65 **A1**

(d) Answer: modulus 225, arg 2.65

Question 14

(a) $|z1| = \sqrt{(-1)^2 + (\sqrt{3})^2} = \sqrt{4} = 2$ **M1**

(a) reference angle $\arctan(\sqrt{3}/1) = \pi/3$, second quadrant, so $\theta = \pi - \pi/3 = 2*\pi/3$ **M1**

(a) $z1 = 2*(\cos(2*\pi/3) + i*\sin(2*\pi/3))$, cao **A1**

(a) Answer: $z1 = 2*(\cos(2*\pi/3) + i*\sin(2*\pi/3))$

(b) $|z2| = \sqrt{2^2 + (-2)^2} = \sqrt{8} = 2*\sqrt{2}$ **M1**

(b) reference angle $\arctan(2/2) = \pi/4$, fourth quadrant, so $\theta = -\pi/4$ **M1**

(b) $z2 = 2*\sqrt{2}*(\cos(-\pi/4) + i*\sin(-\pi/4))$, cao **A1**

(b) Answer: $z2 = 2*\sqrt{2}*(\cos(-\pi/4) + i*\sin(-\pi/4))$

(c) $|z1*z2| = 2*2*\sqrt{2} = 4*\sqrt{2}$ and $\arg(z1*z2) = 2*\pi/3 - \pi/4 = 5*\pi/12$ **M1**

(c) $z1*z2 = 4*\sqrt{2}*(\cos(5*\pi/12) + i*\sin(5*\pi/12))$ **A1**

(c) $|z1/z2| = 2/(2*\sqrt{2}) = \sqrt{2}/2$ and $\arg(z1/z2) = 2*\pi/3 - (-\pi/4) = 11*\pi/12$ **M1**

(c) $z1/z2 = (\sqrt{2}/2)*(\cos(11*\pi/12) + i*\sin(11*\pi/12))$, cao **A1**

(c) Answer: $z1*z2 = 4*\sqrt{2}*(\cos(5*\pi/12) + i*\sin(5*\pi/12))$; $z1/z2 = (\sqrt{2}/2)*(\cos(11*\pi/12) + i*\sin(11*\pi/12))$

(d) uses $11*\pi/12 = \pi - \pi/12$ to write $\cos(11*\pi/12) = -\cos(\pi/12)$ and $\sin(11*\pi/12) = \sin(\pi/12)$ **M1**

(d) $\cos(\pi/12) = (\sqrt{6}+\sqrt{2})/4$ and $\sin(\pi/12) = (\sqrt{6}-\sqrt{2})/4$, found using the addition formulae with $\pi/12 = \pi/3 - \pi/4$ **A1**

(d) substitutes into $(\sqrt{2}/2)*(\cos(11*\pi/12)+i*\sin(11*\pi/12))$ and simplifies the surds, using $\sqrt{2}*\sqrt{6} = 2*\sqrt{3}$ **M1**

(d) $z1/z2 = -(\sqrt{3}+1)/4 + i*(\sqrt{3}-1)/4$, cso **A1**

(d) Answer: $z1/z2 = -(\sqrt{3}+1)/4 + i*(\sqrt{3}-1)/4$

Question 15

(a) $(x-6)^2 + y^2 = x^2 + (y-2)^2$ **M1**

(a) expands and simplifies to -12x + 36 = -4y + 4 **M1**

(a) y = 3x - 8, cso **A1**

(a) Answer: y = 3x - 8

(b) $\tan(\pi/3) = (y-1)/(x-1)$, with x > 1 (and y > 1) since arg = $\pi/3$ requires both components positive **M1**

(b) $y - 1 = \sqrt{3}*(x - 1)$ **A1**

(b) $y = \sqrt{3}*x + 1 - \sqrt{3}$, for x > 1, cso **A1**

(b) Answer: $y = \sqrt{3}*x + 1 - \sqrt{3}$, for x > 1

(c) sets $3x - 8 = \sqrt{3}*x + 1 - \sqrt{3}$ **M1**

(c) solves for x: $x*(3 - \sqrt{3}) = 9 - \sqrt{3}$ **M1**

(c) rationalises to $x = 4 + \sqrt{3}$ **A1**

(c) $y = 3x - 8 = 4 + 3*\sqrt{3}$, giving $z = (4 + \sqrt{3}) + i*(4 + 3*\sqrt{3})$, cao **A1**

(c) Answer: $z = (4 + \sqrt{3}) + i*(4 + 3*\sqrt{3})$

Question 16

(a) writes z = x + i*y and expands $|z-3|=3$ as $(x-3)^2+y^2=9$, giving the Cartesian equation $x^2+y^2 = 6*x$ **M1**

(a) correct Cartesian equation $x^2+y^2 = 6*x$ **A1**

(a) writes $w = 1/z = (x-i*y)/(x^2+y^2)$, rationalising by the conjugate **M1**

(a) substitutes $x^2+y^2=6*x$ to get $w = (x-i*y)/(6*x)$ **A1**

(a) takes the real part: $Re(w) = x/(6*x) = 1/6$, valid whenever x is not 0 **M1**

(a) notes that x=0 on the circle occurs only at the origin, which is excluded, so the result holds for every image point **B1**

(a) 6*Re(w) = 1, cso **A1**

(a) Answer: 6*Re(w) = 1, i.e. Re(w) = 1/6

(b) correct statement: yes, every point of the line is the image of exactly one point on the circle **B1**

(b) valid supporting reason, e.g. that w=1/z is a one-to-one map on the punctured plane (z not equal to 0), and it is a standard result that a circle through the origin (with the origin removed) maps onto the whole of a line under w=1/z **B1**

(b) Answer: Yes: every point on the line 6*Re(w) = 1 is the image of exactly one point on the circle (other than the origin itself, which has no finite image)

(c) combines with part (a): w = 1/6 + i/6 = (1+i)/6 **M1**

(c) z = 1/w = 6/(1+i) **M1**

(c) rationalises: $z = 6*(1-i)/((1+i)*(1-i)) = 6*(1-i)/2 = 3*(1-i)$ **A1**

(c) z = 3 - 3*i, cao **A1**

(c) Answer: z = 3 - 3*i

Question 17

(a) writes $\cos(5*\theta) + i*\sin(5*\theta) = (\cos(\theta) + i*\sin(\theta))^5$ by de Moivre's theorem **M1**

(a) attempts the binomial expansion of the right-hand side **M1**

(a) correct unsimplified expansion: $\cos^5(\theta) + 5i*\cos^4(\theta)\sin(\theta) - 10\cos^3(\theta)\sin^2(\theta) - 10i*\cos^2(\theta)\sin^3(\theta) + 5\cos(\theta)\sin^4(\theta) + i*\sin^5(\theta)$ **A1**

(a) equates real parts: $\cos(5*\theta) = \cos^5(\theta) - 10\cos^3(\theta)\sin^2(\theta) + 5\cos(\theta)\sin^4(\theta)$ **M1**

(a) substitutes $\sin^2(\theta) = 1 - \cos^2(\theta)$ throughout (dependent on the previous method mark) **dM1**

(a) correctly expands and collects terms in $\cos(\theta)$ **A1**

(a) $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$, cso **A1**

(a) Answer: $\cos(5*\theta) = 16*\cos^5(\theta) - 20*\cos^3(\theta) + 5*\cos(\theta)$

(b) uses part (a) to rewrite the equation as $\cos(5*\theta) = 0$ **M1**

(b) sets the general solution $5*\theta = \pi/2 + k*\pi$ **M1**

(b) $\theta = \pi/10 + k*\pi/5$ **M1**

(b) at least three correct values within the range **A1**

(b) all five values correct with no extras: $\theta = \pi/10, 3*\pi/10, \pi/2, 7*\pi/10, 9*\pi/10$ **A1**

(b) Answer: $\theta = \pi/10, 3*\pi/10, \pi/2, 7*\pi/10, 9*\pi/10$

Question 18

(a) $1/z = \cos(\theta) - i*\sin(\theta)$, using de Moivre's theorem with n = -1 (or since $|z|=1$, $1/z$ equals the conjugate of z) **B1**

(a) $z - 1/z = (\cos(\theta)+i*\sin(\theta)) - (\cos(\theta)-i*\sin(\theta)) = 2*i*\sin(\theta)$, cso **B1**

(a) Answer: $z - 1/z = 2*i*\sin(\theta)$

(b) raises both sides of part (a) to the power 5: $(2*i*\sin(\theta))^5 = (z - 1/z)^5$, and attempts the binomial expansion of the right-hand side **M1**

(b) correct expansion $z^5 - 5*z^3 + 10*z - 10*z^{-1} + 5*z^{-3} - z^{-5}$ **A1**

(b) $(2*i)^5 = 32*i$ (since $i^5 = i$), giving $32*i*\sin^5(\theta)$ as required, cso **A1**

(b) Answer: $32*i*\sin^5(\theta) = z^5 - 5*z^3 + 10*z - 10*z^{-1} + 5*z^{-3} - z^{-5}$

(c) groups terms as $(z^5 - z^{-5}) - 5*(z^3 - z^{-3}) + 10*(z - z^{-1})$ **M1**

(c) uses $z^n - z^{-n} = 2*i*\sin(n*\theta)$ for n = 5, 3, 1 **M1**

(c) $32*i*\sin^5(\theta) = 2*i*\sin(5*\theta) - 10*i*\sin(3*\theta) + 20*i*\sin(\theta)$ **A1**

(c) divides both sides by $32*i$ to obtain $\sin^5(\theta) = (1/16)*(\sin(5*\theta) - 5*\sin(3*\theta) + 10*\sin(\theta))$, cso **A1**

(c) Answer: $\sin^5(\theta) = (1/16)*(\sin(5*\theta) - 5*\sin(3*\theta) + 10*\sin(\theta))$

(d) integrates term by term using part (c): integral = $(1/16)*[-\cos(5*\theta)/5 + (5/3)*\cos(3*\theta) - 10*\cos(\theta)]$ **M1**

(d) correct integration (follow through on sign errors) **A1**

(d) substitutes limits $\theta = \pi/2$ and $\theta = 0$, using $\cos(5*\pi/2) = 0$, $\cos(3*\pi/2) = 0$, $\cos(\pi/2) = 0$ and $\cos(0) = 1$ **M1**

(d) exact value = 8/15, cao **A1**

(d) Answer: Integral from 0 to $\pi/2$ of $\sin^5(\theta)\, d(\theta) = 8/15$