1.15 Core Pure: Polar Coordinates and Hyperbolic Functions Depth

Total Marks

EDEXCEL 9FM0 · Calculator allowed · about 220 minutes · Write-in edition

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 This question concerns the inverse hyperbolic function artanh.

(a) Given that y = artanh(x) for |x| < 1, so that x = tanh(y), show that artanh(x) = (1/2)ln((1+x)/(1-x)). **(5)**

(b) Hence solve, for x in the real numbers, the equation 3tanh(x) = 2, giving your answer as an exact single logarithm. **(3)**

(c) Differentiate artanh(2x) with respect to x, simplifying your answer. **(3)**

(Total for Question 1 is 11 marks)

2 This question tests differentiation and integration of hyperbolic functions.

(a) Differentiate $\cosh(1 - 4x^2)$ with respect to x. **(2)**

(b) Differentiate $\ln(\sinh(3x))$ with respect to x, for $x > 0$, simplifying your answer. **(3)**

(c) Differentiate $x^2 \tanh(x)$ with respect to x. **(3)**

(d) Find, in simplest form, the integral of $x \cosh(x^2)$ with respect to x. **(2)**

(e) Find the integral of $\tanh(x)$ with respect to x, giving your answer in the form $\ln(f(x)) + c$. **(3)**

(Total for Question 2 is 13 marks)

3 The curve C has polar equation $r = 4 + 4\cos(\theta)$, for $0 \leq \theta < 2\text{pi}$.

(a) Find the area enclosed by C. **(6)**

(b) Find the area of the region that lies inside C and outside the circle with polar equation $r = 4$. **(7)**

(Total for Question 3 is 13 marks)

4 Solve the equation $33 \cosh x - 27 \sinh x = 21$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $6e^{2x} - 42e^{x} + 60 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 4 is 7 marks)

5 This question is about integrating hyperbolic and related functions.

(a) Find the exact value of the integral from 0 to 25/12 of $1/\sqrt{x^2 + 25}\ dx$, giving your answer as a natural logarithm. **(4)**

(Total for Question 5 is 4 marks)

6 It is given that $\sinh x = 5/12$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 6 is 10 marks)

7 The curve C has equation $y = 8\cosh x - 3\sinh x$.

(a) Find dy/dx and show that C has a stationary point at $x = (1/2)\ln(11/5)$. **(4)**

(b) Find the exact y coordinate of the stationary point, and determine its nature. **(3)**

(Total for Question 7 is 7 marks)

8 The curve C has polar equation $r = 6(1 + \cos \theta)$, for $-\pi < \theta \leq \pi$. The region R is bounded by C, the initial line and the half line $\theta = \pi/2$.

(a) Find the exact area of R. **(4)**

(Total for Question 8 is 4 marks)

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9 Solve the equation $43 \cosh x - 41 \sinh x = 13$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $2e^{2x} - 26e^{x} + 84 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 9 is 7 marks)

10 It is given that sinh x = 15/8, where x is real.

(a) Use a hyperbolic identity to find the exact value of cosh x, and hence tanh x. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of sinh 2x and of cosh 2x. **(3)**

(Total for Question 10 is 10 marks)

11 This question concerns the auxiliary form of expressions such as R cosh(x + a).

(a) Show that 5cosh(x) + 3sinh(x) can be written in the form 4cosh(x + ln(2)). **(6)**

(b) Hence solve, for x in the real numbers, the equation $5\cosh(x) + 3\sinh(x) = 4\sqrt{2}$, giving each solution as an exact logarithm. **(6)**

(Total for Question 11 is 12 marks)

12 The curve C has equation $y = 6\cosh x - 4\sinh x$.

(a) Find $\frac{dy}{dx}$ and show that C has a stationary point at $x = \frac{1}{2}\ln(5)$. **(4)**

(b) Find the exact y coordinate of the stationary point, and determine its nature. **(3)**

(Total for Question 12 is 7 marks)

13 It is given that $\sinh x = \frac{9}{40}$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 13 is 10 marks)

14 Solve the equation $138 \cosh x - 132 \sinh x = 42$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $6e^{2x} - 84e^{x} + 270 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 14 is 7 marks)

15 The curve C has polar equation $r = 10 \cos \theta$, for $-\pi/2 \leq \theta \leq \pi/2$.

(a) Find a Cartesian equation of C. **(3)**

(b) Hence write the equation of C in the form $(x - p)^2 + y^2 = q$, and state the centre and radius of the circle. **(3)**

(Total for Question 15 is 6 marks)

16 The curve C has polar equation $r = 16 \cos \theta$, for $-\pi/2 \leq \theta \leq \pi/2$.

(a) Find a Cartesian equation of C. **(3)**

(b) Hence write the equation of C in the form $(x - p)^2 + y^2 = q$, and state the centre and radius of the circle. **(3)**

(Total for Question 16 is 6 marks)

17 This question concerns the inverse hyperbolic function arcosh.

(a) Given that $y = \text{arcosh}(x)$ for $x \geq 1$, $y \geq 0$, so that $x = \cosh(y)$, show that $\text{arcosh}(x) = \ln(x + \sqrt{x^2 - 1})$. **(5)**

(b) Hence find the exact value of arcosh(13/5), giving your answer as a single logarithm. **(3)**

(c) Hence, by differentiating the result of part (a) directly with respect to x, show that $d/dx(\text{arcosh}(x)) = 1/\sqrt{x^2 - 1}$, for $x > 1$. **(4)**

(Total for Question 17 is 12 marks)

18 A curve has Cartesian equation $(x-1)^2 + (y-1)^2 = 2$.

(a) Show that the polar equation of this curve can be written as $r = 2\cos(\theta) + 2\sin(\theta)$, stating the range of values of θ, $0 \leq \theta < 2\pi$, for which $r \geq 0$. **(5)**

(b) Hence write down the maximum value of r on the curve, and state the value of θ, $0 \leq \theta < 2\pi$, at which this maximum occurs. **(3)**

(Total for Question 18 is 8 marks)

19 This question uses hyperbolic substitutions to derive and apply standard integration results.

(a) Using the substitution $x = 3\cosh(u)$, show that, for $x > 3$, the integral of $1/\sqrt{x^2 - 9}$ with respect to x is $\text{arcosh}(x/3) + c$. **(5)**

(b) Hence find the exact value of the integral, from $x=4$ to $x=6$, of $1/\sqrt{x^2-9}$ with respect to x, giving your answer in terms of logarithms. **(4)**

(c) Using the substitution $x = 5\tanh(u)$, show that, for $-5 < x < 5$, the integral of $1/(25 - x^2)$ with respect to x is $(1/5)\text{artanh}(x/5) + c$. **(5)**

(d) Hence find the exact value of the integral, from $x=0$ to $x=3$, of $1/(25 - x^2)$ with respect to x, giving your answer as a single natural logarithm. **(4)**

(Total for Question 19 is 18 marks)

Mark scheme · 1.15 Core Pure: Polar Coordinates and Hyperbolic Functions Depth

Question 1

(a) write $x = \tanh(y) = (e^{2y}-1)/(e^{2y}+1)$, obtained by multiplying numerator and denominator of $(e^{y}-e^{-y})/(e^{y}+e^{-y})$ by e^{y} **M1**

(a) form the equation $x(e^{2y}+1) = e^{2y} - 1$ **M1**

(a) rearrange to isolate e^{2y}: $x + 1 = e^{2y}(1-x)$ **M1**

(a) $e^{2y} = (1+x)/(1-x)$ **A1**

(a) cso: $y = (1/2)\ln((1+x)/(1-x))$ **A1**

(a) Answer: artanh(x) = (1/2)ln((1+x)/(1-x))

(b) $\tanh(x) = 2/3$, substituted into $x = (1/2)\ln((1+x)/(1-x))$ with $x=2/3$ **M1**

(b) simplify $(1+2/3)/(1-2/3) = (5/3)/(1/3) = 5$ **A1**

(b) cao: $x = (1/2)\ln(5)$ $(= \ln(\sqrt{5}))$ **A1**

(b) Answer: $x = (1/2)\ln(5) = \ln(\sqrt{5})$

(c) use $d/dx[\text{artanh}(u)] = u'/(1-u^2)$ with $u = 2x$ **M1**

(c) correct substitution $u=2x$, $u'=2$ **A1**

(c) cao: $d/dx[\text{artanh}(2x)] = 2/(1-4x^2)$ **A1**

(c) Answer: $d/dx[\text{artanh}(2x)] = 2/(1-4x^2)$

Question 2

(a) apply the chain rule: derivative of cosh(u) is sinh(u) times u', with $u = 1-4x^2$, $u' = -8x$ **M1**

(a) cao: $-8x \sinh(1-4x^2)$ **A1**

(a) Answer: $-8x \sinh(1 - 4x^2)$

(b) apply the chain rule: (1/sinh(3x)) times derivative of sinh(3x) **M1**

(b) derivative of sinh(3x) = 3cosh(3x) used correctly **A1**

(b) cao: simplify (3cosh(3x))/sinh(3x) = 3coth(3x) **A1**

(b) Answer: 3coth(3x)

(c) apply the product rule u'v + uv' with $u = x^2$, $v = \tanh(x)$ **M1**

(c) use derivative of $\tanh(x) = \text{sech}^2(x)$ correctly **A1**

(c) cao: $2x \tanh(x) + x^2 \text{sech}^2(x)$ **A1**

(c) Answer: $2x \tanh(x) + x^2 \text{sech}^2(x)$

(d) recognise $x \cosh(x^2)$ as (1/2) times the derivative of $\sinh(x^2)$ **M1**

(d) cao: $(1/2)\sinh(x^2) + c$ **A1**

(d) Answer: $(1/2)\sinh(x^2) + c$

(e) write $\tanh(x) = \sinh(x)/\cosh(x)$ and recognise the numerator as the derivative of the denominator **M1**

(e) integral form $\ln|\cosh(x)| + c$ **A1**

(e) cao: since $\cosh(x) > 0$ for all real x, this simplifies to $\ln(\cosh(x)) + c$ **A1**

(e) Answer: $\ln(\cosh(x)) + c$

Question 3

(a) Area = (1/2) times the integral of r^2 with respect to θ, with limits 0 to 2pi **M1**

(a) expand $(4+4\cos(\theta))^2 = 16 + 32\cos(\theta) + 16\cos^2(\theta)$ **M1**

(a) use $\cos^2(\theta) = (1+\cos(2\theta))/2$ to rewrite the integrand **M1**

(a) correct integral: (1/2) times the integral of $(16 + 32\cos(\theta) + 8 + 8\cos(2\theta))$ dtheta **A1**

(a) integrate term by term and substitute the limits 0 to 2pi (the cos(θ) and cos(2theta) terms integrate to zero over a full period) **M1**

(a) cao: Area = 24pi **A1**

(a) Answer: Area = 24pi

(b) find the points of intersection: $4+4\cos(\theta) = 4$ gives $\cos(\theta) = 0$, so $\theta = +\text{-}\pi/2$ **M1**

(b) set up Area = (1/2) times the integral, with limits $-\pi/2$ to $\pi/2$, of $[(4+4\cos(\theta))^2 - 4^2]$ dtheta **M1**

(b) expand the integrand to $32\cos(\theta) + 16\cos^2(\theta)$ **M1**

(b) use $\cos^2(\theta) = (1+\cos(2\theta))/2$ to write the integrand as $32\cos(\theta) + 8 + 8\cos(2\theta)$ **A1**

(b) integrate to obtain $32\sin(\theta) + 8\theta + 4\sin(2\theta)$ **dM1**

(b) correct evaluation between the limits gives $64 + 8\pi$ **A1**

(b) cao: Area = $(1/2)(64+8\pi) = 32 + 4\pi$ **A1**

(b) Answer: Area = $32 + 4\pi$

Question 4

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ **M1**

(a) multiplies through by e^x and collects terms **M1**

(a) $6e^{2x} - 42e^x + 60 = 0$ (cso) **A1**

(a) Answer: $6e^{2x} - 42e^x + 60 = 0$

(b) treats as a quadratic in e^x and factorises **M1**

(b) $e^x = 2$ or $e^x = 5$ **A1**

(b) takes natural logs **M1**

(b) $x = \ln 2$ and $x = \ln 5$ **A1**

(b) Answer: $x = \ln 2$, $x = \ln 5$

Question 5

(a) uses the standard integral: arsinh(x/5) **M1**

(a) $\text{arsinh}(5/12) = \ln(5/12 + \sqrt{1 + 25/144})$ **A1**

(a) $\sqrt{1 + 25/144} = 13/12$ **A1**

(a) $\ln(3/2)$ **A1**

(a) Answer: $\ln(3/2)$

Question 6

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 25/144 = 169/144$ **A1**

(a) $\cosh x = 13/12$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 5/13$ **A1**

(a) Answer: cosh x = 13/12, tanh x = 5/13

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 3/2$ **A1**

(b) $x = \ln(3/2)$ **A1**

(b) Answer: x = ln(3/2)

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 65/72$ **A1**

(c) $\cosh 2x = 97/72$ **A1**

(c) Answer: sinh 2x = 65/72, cosh 2x = 97/72

Question 7

(a) $dy/dx = 8 \sinh x - 3 \cosh x$ **M1**

(a) sets $dy/dx = 0$: $\tanh x = 3/8$ **M1**

(a) $x = \text{artanh}(3/8) = (1/2) \ln((1 + 3/8)/(1 - 3/8))$ **A1**

(a) $= (1/2) \ln(11/5)$ (cso) **A1**

(a) Answer: x = (1/2) ln(11/5)

(b) substitutes: $\cosh x = 8/\sqrt{55}$ etc, or writes $y = \sqrt{55} \cosh(x - c)$ **M1**

(b) $y = \sqrt{55}$ **A1**

(b) $d^2y/dx^2 = y > 0$ at the stationary point, so a minimum **B1**

(b) Answer: $y = \sqrt{55}$ (minimum)

Question 8

(a) area $= (1/2) \int_0^{\pi/2} r^2 \, d\theta$ **M1**

(a) uses $\cos^2 \theta = (1 + \cos 2\text{theta})/2$ and integrates **M1**

(a) substitutes the limits 0 and $\pi/2$ **M1**

(a) $36(3\text{pi}/8 + 1)$ **A1**

(a) Answer: 36(3pi/8 + 1)

Question 9

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ **M1**

(a) multiplies through by e^x and collects terms **M1**

(a) $2e^{2x} - 26e^x + 84 = 0$ (cso) **A1**

(a) Answer: $2e^{2x} - 26e^x + 84 = 0$

(b) treats as a quadratic in e^x and factorises **M1**

(b) $e^x = 6$ or $e^x = 7$ **A1**

(b) takes natural logs **M1**

(b) $x = \ln 6$ and $x = \ln 7$ **A1**

(b) Answer: x = ln 6, x = ln 7

Question 10

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 225/64 = 289/64$ **A1**

(a) $\cosh x = 17/8$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 15/17$ **A1**

(a) Answer: cosh x = 17/8, tanh x = 15/17

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 4$ **A1**

(b) $x = \ln 4$ **A1**

(b) Answer: x = ln 4

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 255/32$ **A1**

(c) $\cosh 2x = 257/32$ **A1**

(c) Answer: sinh 2x = 255/32, cosh 2x = 257/32

Question 11

(a) write $R \cosh(x+a) = R \cosh(a)\cosh(x) + R \sinh(a)\sinh(x)$ and match coefficients: $R \cosh(a) = 5$, $R \sinh(a) = 3$ **M1**

(a) add and subtract the two equations, using $e^a = \cosh(a)+\sinh(a)$, to get $R e^a = 8$ and $R e^{-a} = 2$ **M1**

(a) multiply these two results: $R^2 = 16$ **M1**

(a) $R = 4$ ($R > 0$) **A1**

(a) divide the two results: $e^{2a} = 4$, so $e^a = 2$ ($e^a > 0$) **M1**

(a) cso: $a = \ln(2)$, giving $5\cosh(x)+3\sinh(x) = 4\cosh(x+\ln(2))$ **A1**

(a) Answer: 5cosh(x) + 3sinh(x) = 4cosh(x + ln(2))

(b) use the result of part (a) to write $4\cosh(x+\ln(2)) = 4\sqrt{2}$ **M1**

(b) $\cosh(x+\ln(2)) = \sqrt{2}$ **M1**

(b) use $\text{arcosh}(\sqrt{2}) = \ln(\sqrt{2} + \sqrt{2-1}) = \ln(\sqrt{2}+1)$, so $x+\ln(2) = +\text{-}\ln(\sqrt{2}+1)$ **M1**

(b) both values of $x+\ln(2)$ correctly identified **A1**

(b) rearrange each to make x the subject: $x = \ln(\sqrt{2}+1) - \ln(2)$ or $x = -\ln(\sqrt{2}+1) - \ln(2)$ **M1**

(b) cao: $x = \ln((\sqrt{2}+1)/2)$ or $x = -\ln(2\sqrt{2}+2)$ (oe unsimplified forms accepted) **A1**

(b) Answer: $x = \ln((\sqrt{2}+1)/2)$ or $x = -\ln(2\sqrt{2}+2)$

Question 12

(a) $dy/dx = 6 \sinh x - 4 \cosh x$ **M1**

(a) sets $dy/dx = 0$: $\tanh x = 2/3$ **M1**

(a) $x = \text{artanh}(2/3) = (1/2) \ln((1 + 2/3)/(1 - 2/3))$ **A1**

(a) $= (1/2) \ln(5)$ (cso) **A1**

(a) Answer: x = (1/2) ln(5)

(b) substitutes: $\cosh x = 6/\sqrt{20}$ etc, or writes $y = \sqrt{20} \cosh(x - c)$ **M1**

(b) $y = 2\sqrt{5}$ **A1**

(b) $d^2y/dx^2 = y > 0$ at the stationary point, so a minimum **B1**

(b) Answer: $y = 2\sqrt{5}$ (minimum)

Question 13

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 81/1600 = 1681/1600$ **A1**

(a) $\cosh x = 41/40$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 9/41$ **A1**

(a) Answer: $\cosh x = 41/40$, $\tanh x = 9/41$

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 5/4$ **A1**

(b) $x = \ln(5/4)$ **A1**

(b) Answer: $x = \ln(5/4)$

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 369/800$ **A1**

(c) $\cosh 2x = 881/800$ **A1**

(c) Answer: $\sinh 2x = 369/800$, $\cosh 2x = 881/800$

Question 14

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ **M1**

(a) multiplies through by e^x and collects terms **M1**

(a) $6e^{2x} - 84e^x + 270 = 0$ (cso) **A1**

(a) Answer: $6e^{2x} - 84e^x + 270 = 0$

(b) treats as a quadratic in e^x and factorises **M1**

(b) $e^x = 5$ or $e^x = 9$ **A1**

(b) takes natural logs **M1**

(b) $x = \ln 5$ and $x = \ln 9$ **A1**

(b) Answer: $x = \ln 5$, $x = \ln 9$

Question 15

(a) multiplies by r: $r^2 = 10 r \cos\theta$ **M1**

(a) uses $r^2 = x^2 + y^2$ and $r\cos\theta = x$ **M1**

(a) $x^2 + y^2 = 10x$ **A1**

(a) Answer: $x^2 + y^2 = 10x$

(b) completes the square **M1**

(b) $(x - 5)^2 + y^2 = 25$ **A1**

(b) centre (5, 0), radius 5 **A1**

(b) Answer: $(x - 5)^2 + y^2 = 25$; centre (5, 0), radius 5

Question 16

(a) multiplies by r: $r^2 = 16 r \cos\theta$ **M1**

(a) uses $r^2 = x^2 + y^2$ and $r\cos\theta = x$ **M1**

(a) $x^2 + y^2 = 16x$ **A1**

(a) Answer: $x^2 + y^2 = 16x$

(b) completes the square **M1**

(b) $(x - 8)^2 + y^2 = 64$ **A1**

(b) centre (8, 0), radius 8 **A1**

(b) Answer: $(x - 8)^2 + y^2 = 64$; centre (8, 0), radius 8

Question 17

(a) state $x = \cosh(y)$ **M1**

(a) use $\cosh^2(y) - \sinh^2(y) = 1$ to write $\sinh(y) = \sqrt{x^2 - 1}$ **M1**

(a) justify the positive square root, since $y \geq 0$ implies $\sinh(y) \geq 0$ **A1**

(a) use $e^y = \cosh(y) + \sinh(y) = x + \sqrt{x^2 - 1}$ **M1**

(a) cso: $y = \ln(x + \sqrt{x^2-1})$, so $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1})$ **A1**

(a) Answer: $\operatorname{arcosh}(x) = \ln(x + \sqrt{x^2-1})$

(b) substitute $x = 13/5$ into $\ln(x + \sqrt{x^2-1})$ **M1**

(b) simplify $\sqrt{169/25 - 1} = \sqrt{144/25} = 12/5$ **A1**

(b) cao: $\operatorname{arcosh}(13/5) = \ln(5)$ **A1**

(b) Answer: $\operatorname{arcosh}(13/5) = \ln(5)$

(c) differentiate $\ln(x+\sqrt{x^2-1})$ using the chain rule: $(1/(x+\sqrt{x^2-1})) * (1 + x/\sqrt{x^2-1})$ **M1**

(c) combine the bracket over a common denominator: $(\sqrt{x^2-1}+x)/\sqrt{x^2-1}$ **M1**

(c) cancel the common factor $(x+\sqrt{x^2-1})$ from numerator and denominator **A1**

(c) cso: $d/dx(\operatorname{arcosh}(x)) = 1/\sqrt{x^2-1}$ **A1**

(c) Answer: $d/dx(\operatorname{arcosh}(x)) = 1/\sqrt{x^2-1}$, for $x > 1$

Question 18

(a) expand $(x-1)^2 + (y-1)^2 = 2$ to $x^2 + y^2 - 2x - 2y = 0$ **M1**

(a) substitute $x^2+y^2 = r^2$, $x = r\cos(\theta)$, $y = r\sin(\theta)$: $r^2 - 2r\cos(\theta) - 2r\sin(\theta) = 0$ **M1**

(a) divide by r (r not identically 0): $r = 2\cos(\theta) + 2\sin(\theta)$, cso **M1**

(a) find where $r = 0$: $\tan(\theta) = -1$, giving $\theta = 3\pi/4$ or $\theta = 7\pi/4$ **A1**

(a) cao: $r \geq 0$ for $0 \leq \theta \leq 3\pi/4$ or $7\pi/4 \leq \theta < 2\pi$ **A1**

(a) Answer: $r = 2\cos(\theta) + 2\sin(\theta)$; $r \geq 0$ for $0 \leq \theta \leq 3\pi/4$ or $7\pi/4 \leq \theta < 2\pi$

(b) express $r = 2\cos(\theta) + 2\sin(\theta)$ in the form $R\cos(\theta - \alpha)$ with $R = 2\sqrt{2}$, $\tan(\alpha) = 1$ **M1**

(b) identify that the maximum value of r is R, occurring when $\theta = \alpha = \pi/4$ **M1**

(b) cao: maximum $r = 2\sqrt{2}$, at $\theta = \pi/4$ **A1**

(b) Answer: Maximum $r = 2\sqrt{2}$, at $\theta = \pi/4$

Question 19

(a) substitute $x = 3\cosh(u)$, $dx = 3\sinh(u)\,du$ **M1**

(a) use $\cosh^2(u) - 1 = \sinh^2(u)$ to write $\sqrt{x^2-9} = \sqrt{9\cosh^2(u)-9} = 3\sinh(u)$ **M1**

(a) integral simplifies to the integral of 1 with respect to u **A1**

(a) integrate to get u + c and use $u = \text{arcosh}(x/3)$ **M1**

(a) cso: integral of $1/\sqrt{x^2-9}\,dx = \text{arcosh}(x/3) + c$ **A1**

(a) Answer: arcosh(x/3) + c

(b) substitute the limits into arcosh(x/3): arcosh(2) - arcosh(4/3) **M1**

(b) use $\text{arcosh}(2) = \ln(2 + \sqrt{4-1}) = \ln(2+\sqrt{3})$ **M1**

(b) use $\text{arcosh}(4/3) = \ln(4/3 + \sqrt{16/9-1}) = \ln((4+\sqrt{7})/3)$ **A1**

(b) cao: value $= \ln(2+\sqrt{3}) - \ln((4+\sqrt{7})/3)$, oe (e.g. a single combined logarithm) **A1**

(b) Answer: $\ln(2+\sqrt{3}) - \ln((4+\sqrt{7})/3)$, approx 0.522 (3 s.f.)

(c) substitute $x = 5\tanh(u)$, $dx = 5\text{sech}^2(u)\,du$ **M1**

(c) use $1 - \tanh^2(u) = \text{sech}^2(u)$ to write $25-x^2 = 25(1-\tanh^2(u)) = 25\text{sech}^2(u)$ **M1**

(c) integral simplifies to the integral of 1/5 with respect to u **A1**

(c) integrate to get u/5 + c and use $u = \text{artanh}(x/5)$ **M1**

(c) cso: integral of $1/(25-x^2)\,dx = (1/5)\text{artanh}(x/5) + c$ **A1**

(c) Answer: (1/5)artanh(x/5) + c

(d) substitute the limits into (1/5)artanh(x/5): (1/5)[artanh(3/5) - artanh(0)] **M1**

(d) artanh(0) = 0, so the value is (1/5)artanh(3/5) **A1**

(d) use $\text{artanh}(3/5) = (1/2)\ln((1+3/5)/(1-3/5)) = (1/2)\ln(4) = \ln(2)$ **M1**

(d) cao: (1/5)ln(2) **A1**

(d) Answer: (1/5)ln(2)