1.7 Hyperbolic Functions

EDEXCEL 9FM0 · Calculator allowed · about 230 minutes · Write-in edition

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 This question concerns the graphs of the hyperbolic functions.

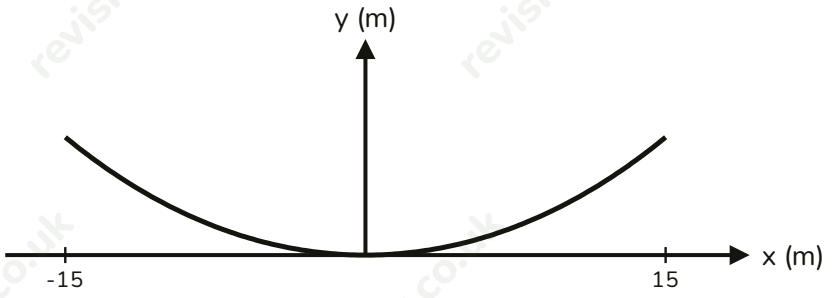
(a) Sketch the graph of $y = \cosh x$, marking the y-intercept and stating the coordinates and nature of any turning point. **(3)**

(b) Sketch the graph of $y = \tanh x$, stating the equations of its horizontal asymptotes. **(2)**

(c) With reference to the graph of $y = \sinh x$, explain why the equation $\sinh x = k$ has exactly one real solution for every real number k. **(2)**

(Total for Question 1 is 7 marks)

2 An engineer, Priya, models a cable hanging between two supports of equal height as a catenary with equation $y = 20\cosh(x/20) - 20$, for $-15 \leq x \leq 15$, where x and y are measured in metres and $x = 0$ at the lowest point of the cable.



(a) Find the sag of the cable, that is, the value of y at $x = 15$, giving your answer in metres to 3 significant figures. **(3)**

(b) Find dy/dx and hence find the gradient of the cable at $x = 15$, giving your answer to 3 significant figures. **(3)**

(c) Using the arc length formula L = integral from x=-15 to x=15 of $\sqrt{1 + (dy/dx)^2}$ dx, and the identity $1 + \sinh^2(t) = \cosh^2(t)$, show that the total length of cable between the two supports is 32.9 m, correct to 3 significant figures. **(5)**

(Total for Question 2 is 11 marks)

3 Solve the equation $32 \cosh x - 28 \sinh x = 16$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $4e^{2x} - 32e^{x} + 60 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 3 is 7 marks)

4 This question is about integrating hyperbolic and related functions.

(a) Find the exact value of the integral from 0 to 9/4 of $1/\sqrt{x^2 + 9}\ dx$, giving your answer as a natural logarithm. **(4)**

(Total for Question 4 is 4 marks)

5 It is given that $\sinh x = \frac{48}{55}$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 5 is 10 marks)

6 Solve the equation $21 \cosh x - 15 \sinh x = 15$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $6e^{2x} - 30e^{x} + 36 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 6 is 7 marks)

7 The curve C has equation $y = 5\cosh x - 1\sinh x$.

(a) Find dy/dx and show that C has a stationary point at $x = (1/2)\ln(3/2)$. **(4)**

(b) Find the exact y coordinate of the stationary point, and determine its nature. **(3)**

(Total for Question 7 is 7 marks)

8 It is given that $\sinh x = 8/15$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 8 is 10 marks)

9 The curve C has equation $y = 2\cosh x - 1\sinh x$.

(a) Find dy/dx and show that C has a stationary point at $x = (1/2)\ln(3)$. **(4)**

(b) Find the exact y coordinate of the stationary point, and determine its nature. **(3)**

(Total for Question 9 is 7 marks)

10 This question is about integrating hyperbolic and related functions.

(a) Find the exact value of the integral from 0 to 5/6 of $1/\sqrt{x^2 + 4}\ dx$, giving your answer as a natural logarithm. **(4)**

(Total for Question 10 is 4 marks)

11 It is given that $\sinh x = \frac{11}{60}$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 11 is 10 marks)

12 The curve C has equation $y = 5\cosh x - 3\sinh x$.

(a) Find dy/dx and show that C has a stationary point at $x = (1/2)\ln(4)$. **(4)**

(b) Find the exact y coordinate of the stationary point, and determine its nature. **(3)**

(Total for Question 12 is 7 marks)

13 This question is about integrating hyperbolic and related functions.

(a) Find the exact value of the integral from 0 to 15/4 of $1/\sqrt{x^2 + 25}\ dx$, giving your answer as a natural logarithm. **(4)**

(Total for Question 13 is 4 marks)

14 It is given that $\sinh x = 4/3$, where x is real.

(a) Use a hyperbolic identity to find the exact value of $\cosh x$, and hence $\tanh x$. **(4)**

(b) Find the exact value of x, giving your answer as a natural logarithm. **(3)**

(c) Find the exact value of $\sinh 2x$ and of $\cosh 2x$. **(3)**

(Total for Question 14 is 10 marks)

15 Solve the equation $9 \cosh x - 7 \sinh x = 6$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $2e^{2x} - 12e^{x} + 16 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 15 is 7 marks)

16 Solve the equation $146 \cosh x - 142 \sinh x = 34$, giving your answers as exact logarithms.

(a) Show that the equation can be written as $4e^{2x} - 68e^{x} + 288 = 0$. **(3)**

(b) Hence solve the equation. **(4)**

(Total for Question 16 is 7 marks)

17 This question is about the inverse hyperbolic function arsinh x, defined for all real x.

(a) Let $y = \text{arsinh } x$, so that $x = \sinh y$. By writing this equation in exponential form, show that $e^{2y} - 2x\,e^{y} - 1 = 0$, and hence show that $\text{arsinh } x = \ln(x + \sqrt{x^2+1})$ for all real x. **(5)**

(b) Hence find, in the form $\ln(a + \sqrt{b})$ where a and b are integers, the exact value of arsinh(2). **(2)**

(c) Solve the equation $\text{arsinh}(x) = \ln 3$, giving your answer as an exact fraction. **(3)**

(Total for Question 17 is 10 marks)

18 This question is about differentiating hyperbolic and inverse hyperbolic functions.

(a) Differentiate $y = \sinh(3x^2 - 1)$ with respect to x. **(2)**

(b) Find dy/dx where $y = \ln(\cosh 2x)$, giving your answer in terms of tanh 2x. **(3)**

(c) Given that $y = \text{arcosh}(3x)$, for $x > 1/3$, show by implicit differentiation that $dy/dx = 3/\sqrt{9x^2-1}$. **(3)**

(d) Find the equation of the tangent to the curve $y = \text{arsinh}(x/2)$ at the point where $x = 0$, giving your answer in the form $y = mx$. **(4)**

(Total for Question 18 is 12 marks)

19 This question is about integrating functions involving hyperbolic functions.

(a) Find, in terms of x, the integral: integral of sinh(4x - 1) dx. **(2)**

(b) Find, in terms of arsinh, the integral: integral of $1/\sqrt{x^2+9}$ dx. **(3)**

(c) Using the substitution x = 4 sinh t, show that integral of $1/\sqrt{x^2+16}$ dx = arsinh(x/4) + c. **(5)**

(d) Hence evaluate exactly the integral from $x = 0$ to $x = 4$ of $1/\sqrt{x^2+16}$ dx, giving your answer as a single natural logarithm. **(2)**

(Total for Question 19 is 12 marks)

20 This question builds towards solving a hyperbolic equation using the double angle formula for cosh.

(a) Using the exponential definitions of cosh x and sinh x, show that cosh(x+y) = cosh x cosh y + sinh x sinh y. **(4)**

(b) Hence show that cosh 2x = 2 $\cosh^2(x)$ - 1. **(3)**

(c) Hence, or otherwise, solve the equation cosh 2x = 5 cosh x - 4, giving each solution exactly in terms of natural logarithms. **(5)**

(Total for Question 20 is 12 marks)

21 This question is about integration leading to the logarithmic form of artanh.

(a) Using the standard result integral of $1/(a^2-x^2)\ dx = (1/a)\ \text{artanh}(x/a) + c$ and the logarithmic form $\text{artanh}(u) = (1/2)\ \ln((1+u)/(1-u))$, show that, for $-3 < x < 3$, integral of $1/(9-x^2)\ dx = (1/6)\ \ln((3+x)/(3-x)) + c$. **(4)**

(b) Hence find the exact value of the integral from $x = 0$ to $x = 1$ of $1/(9-x^2)\ dx$, giving your answer in the form $(1/6)\ \ln k$, stating the value of k. **(3)**

(Total for Question 21 is 7 marks)

22 Consider the equation $3\sinh^2(x) - 5\cosh x + 3 = 0$.

(a) Show that the equation $3\sinh^2(x) - 5\cosh x + 3 = 0$ can be written as $\cosh x(3\cosh x - 5) = 0$. **(3)**

(b) Hence solve the equation $3\sinh^2(x) - 5\cosh x + 3 = 0$, explaining why one of the values of $\cosh x$ found must be rejected, and giving your final answer(s) as exact logarithms. **(5)**

(Total for Question 22 is 8 marks)

Mark scheme · 1.7 Hyperbolic Functions

Question 1

(a) correct shape: even function, symmetrical about the y-axis, $y \geq 1$ for all x — **B1**

(a) y-intercept marked at (0, 1) — **B1**

(a) minimum turning point identified at (0, 1) — **B1**

(a) Answer: U-shaped curve symmetrical about the y-axis, minimum at (0,1), y-intercept (0,1), no x-intercepts

(b) correct S-shaped, increasing, odd-function curve through the origin — **B1**

(b) asymptotes $y = 1$ and $y = -1$ stated — **B1**

(b) Answer: $y = 1$ and $y = -1$

(c) sinh x is a strictly increasing function for all real x, with range all of R — **B1**

(c) hence the horizontal line $y = k$ meets the graph exactly once for any real k — **B1**

(c) Answer: sinh x is a strictly increasing bijection from R to R, so $y = k$ meets the graph exactly once for every real k

Question 2

(a) substitute $x = 15$ into $y = 20\cosh(x/20) - 20$ — **M1**

(a) evaluate cosh(0.75) using a calculator — **M1**

(a) $y = 5.89$ m (awrt 5.89), cao — **A1**

(a) Answer: 5.89 m

(b) differentiate: $dy/dx = 20 \cdot (1/20) \sinh(x/20) = \sinh(x/20)$ — **M1**

(b) substitute $x = 15$ and evaluate sinh(0.75) — **M1**

(b) gradient = 0.822 (awrt 0.822), cao — **A1**

(b) Answer: $dy/dx = \sinh(x/20)$; at $x = 15$, gradient is awrt 0.822

(c) state L = integral from -15 to 15 of $\sqrt{1+\sinh^2(x/20)}$ dx, using their dy/dx from part (b) — **M1**

(c) use $1 + \sinh^2(x/20) = \cosh^2(x/20)$ to simplify the integrand to $\cosh(x/20)$ — **M1**

(c) integrate: integral of $\cosh(x/20)\,dx = 20\sinh(x/20)$ — **M1**

(c) evaluate limits: $20\sinh(0.75) - 20\sinh(-0.75) = 40\sinh(0.75)$ — **A1**

(c) L = 40(0.82232) = 32.9 m (awrt 32.9), cso — **A1**

(c) Answer: L = 32.9 m

Question 3

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ — **M1**

(a) multiplies through by e^x and collects terms — **M1**

(a) $4e^{2x} - 32e^x + 60 = 0$ (cso) — **A1**

(a) Answer: $4e^{2x} - 32e^x + 60 = 0$

(b) treats as a quadratic in e^x and factorises — **M1**

(b) $e^x = 3$ or $e^x = 5$ — **A1**

(b) takes natural logs — **M1**

(b) $x = \ln 3$ and $x = \ln 5$ — **A1**

(b) Answer: $x = \ln 3$, $x = \ln 5$

Question 4

(a) uses the standard integral: arsinh(x/3) — **M1**

(a) $\text{arsinh}(3/4) = \ln(3/4 + \sqrt{1 + 9/16})$ — **A1**

(a) $\sqrt{1 + 9/16} = 5/4$ — **A1**

(a) $\ln 2$ — **A1**

(a) Answer: $\ln 2$

Question 5

(a) uses $\cosh^2 x - \sinh^2 x = 1$ — **M1**

(a) $\cosh^2 x = 1 + 2304/3025 = 5329/3025$ — **A1**

(a) $\cosh x = 73/55$ (positive root, since $\cosh x \geq 1$) — **A1**

(a) $\tanh x = 48/73$ — **A1**

(a) Answer: $\cosh x = 73/55$, $\tanh x = 48/73$

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh — **M1**

(b) $e^x = \sinh x + \cosh x = 11/5$ — **A1**

(b) $x = \ln(11/5)$ — **A1**

(b) Answer: $x = \ln(11/5)$

(c) uses $\sinh 2x = 2\sinh x \cosh x$ and $\cosh 2x = 1 + 2\sinh^2 x$ — **M1**

(c) $\sinh 2x = 7008/3025$ — **A1**

(c) $\cosh 2x = 7633/3025$ — **A1**

(c) Answer: $\sinh 2x = 7008/3025$, $\cosh 2x = 7633/3025$

Question 6

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ — **M1**

(a) multiplies through by e^x and collects terms — **M1**

(a) $6e^{2x} - 30e^x + 36 = 0$ (cso) — **A1**

(a) Answer: $6e^{2x} - 30e^x + 36 = 0$

(b) treats as a quadratic in e^x and factorises — **M1**

(b) $e^x = 2$ or $e^x = 3$ — **A1**

(b) takes natural logs — **M1**

(b) $x = \ln 2$ and $x = \ln 3$ — **A1**

(b) Answer: $x = \ln 2$, $x = \ln 3$

Question 7

(a) $dy/dx = 5\sinh x - 1\cosh x$ — **M1**

(a) sets $dy/dx = 0$: $\tanh x = 1/5$ — **M1**

(a) $x = \text{artanh}(1/5) = (1/2)\ln((1 + 1/5)/(1 - 1/5))$ — **A1**

(a) $= (1/2)\ln(3/2)$ (cso) — **A1**

(a) Answer: $x = (1/2)\ln(3/2)$

(b) substitutes: $\cosh x = 5/\sqrt{24}$ etc, or writes $y = \sqrt{24}\cosh(x - c)$ — **M1**

(b) $y = 2\sqrt{6}$ — **A1**

(b) $d^2y/dx^2 = y > 0$ at the stationary point, so a minimum — **B1**

(b) Answer: $y = 2\sqrt{6}$ (minimum)

Question 8

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 64/225 = 289/225$ **A1**

(a) $\cosh x = 17/15$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 8/17$ **A1**

(a) Answer: $\cosh x = 17/15$, $\tanh x = 8/17$

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 5/3$ **A1**

(b) $x = \ln(5/3)$ **A1**

(b) Answer: $x = \ln(5/3)$

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 272/225$ **A1**

(c) $\cosh 2x = 353/225$ **A1**

(c) Answer: $\sinh 2x = 272/225$, $\cosh 2x = 353/225$

Question 9

(a) $dy/dx = 2 \sinh x - 1 \cosh x$ **M1**

(a) sets $dy/dx = 0$: $\tanh x = 1/2$ **M1**

(a) $x = \text{artanh}(1/2) = (1/2) \ln((1 + 1/2)/(1 - 1/2))$ **A1**

(a) $= (1/2) \ln(3)$ (cso) **A1**

(a) Answer: $x = (1/2) \ln(3)$

(b) substitutes: $\cosh x = 2/\sqrt{3}$ etc, or writes $y = \sqrt{3} \cosh(x - c)$ **M1**

(b) $y = \sqrt{3}$ **A1**

(b) $d^2y/dx^2 = y > 0$ at the stationary point, so a minimum **B1**

(b) Answer: $y = \sqrt{3}$ (minimum)

Question 10

(a) uses the standard integral: $\text{arsinh}(x/2)$ **M1**

(a) $\text{arsinh}(5/12) = \ln(5/12 + \sqrt{1 + 25/144})$ **A1**

(a) $\sqrt{1 + 25/144} = 13/12$ **A1**

(a) $\ln(3/2)$ **A1**

(a) Answer: $\ln(3/2)$

Question 11

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 121/3600 = 3721/3600$ **A1**

(a) $\cosh x = 61/60$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 11/61$ **A1**

(a) Answer: $\cosh x = 61/60$, $\tanh x = 11/61$

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 6/5$ **A1**

(b) $x = \ln(6/5)$ **A1**

(b) Answer: $x = \ln(6/5)$

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 671/1800$ **A1**

(c) $\cosh 2x = 1921/1800$ **A1**

(c) Answer: $\sinh 2x = 671/1800$, $\cosh 2x = 1921/1800$

Question 12

(a) $dy/dx = 5 \sinh x - 3 \cosh x$ **M1**

(a) sets $dy/dx = 0$: $\tanh x = 3/5$ **M1**

(a) $x = \text{artanh}(3/5) = (1/2) \ln((1 + 3/5)/(1 - 3/5))$ **A1**

(a) $= (1/2) \ln(4)$ (cso) **A1**

(a) Answer: $x = (1/2) \ln(4)$

(b) substitutes: $\cosh x = 5/\sqrt{16}$ etc, or writes $y = \sqrt{16} \cosh(x - c)$ **M1**

(b) $y = 4$ **A1**

(b) $d^2y/dx^2 = y > 0$ at the stationary point, so a minimum **B1**

(b) Answer: $y = 4$ (minimum)

Question 13

(a) uses the standard integral: $\text{arsinh}(x/5)$ **M1**

(a) $\text{arsinh}(3/4) = \ln(3/4 + \sqrt{1 + 9/16})$ **A1**

(a) $\sqrt{1 + 9/16} = 5/4$ **A1**

(a) $\ln 2$ **A1**

(a) Answer: $\ln 2$

Question 14

(a) uses $\cosh^2 x - \sinh^2 x = 1$ **M1**

(a) $\cosh^2 x = 1 + 16/9 = 25/9$ **A1**

(a) $\cosh x = 5/3$ (positive root, since $\cosh x \geq 1$) **A1**

(a) $\tanh x = 4/5$ **A1**

(a) Answer: $\cosh x = 5/3$, $\tanh x = 4/5$

(b) uses the definition $\sinh x = (e^x - e^{-x})/2$ or the logarithmic form of arsinh **M1**

(b) $e^x = \sinh x + \cosh x = 3$ **A1**

(b) $x = \ln 3$ **A1**

(b) Answer: $x = \ln 3$

(c) uses $\sinh 2x = 2 \sinh x \cosh x$ and $\cosh 2x = 1 + 2 \sinh^2 x$ **M1**

(c) $\sinh 2x = 40/9$ **A1**

(c) $\cosh 2x = 41/9$ **A1**

(c) Answer: $\sinh 2x = 40/9$, $\cosh 2x = 41/9$

Question 15

(a) writes $\cosh x = (e^x + e^{-x})/2$ and $\sinh x = (e^x - e^{-x})/2$ **M1**

(a) multiplies through by e^x and collects terms **M1**

(a) $2e^{2x} - 12e^x + 16 = 0$ (cso) **A1**

(a) Answer: $2e^{2x} - 12e^x + 16 = 0$

(b) treats as a quadratic in e^x and factorises **M1**

(b) $e^x = 2$ or $e^x = 4$ **A1**

(b) takes natural logs **M1**

(b) $x = \ln 2$ and $x = \ln 4$ **A1**

(b) Answer: $x = \ln 2$, $x = \ln 4$

Question 16

(a) writes cosh x = $(e^x + e^{-x})/2$ and sinh x = $(e^x - e^{-x})/2$ **M1**

(a) multiplies through by e^x and collects terms **M1**

(a) $4e^{2x} - 68e^x + 288 = 0$ (cso) **A1**

(a) Answer: $4e^{2x} - 68e^x + 288 = 0$

(b) treats as a quadratic in e^x and factorises **M1**

(b) $e^x = 8$ or $e^x = 9$ **A1**

(b) takes natural logs **M1**

(b) x = ln 8 and x = ln 9 **A1**

(b) Answer: x = ln 8, x = ln 9

Question 17

(a) write x = sinh y = $(e^y - e^{-y})/2$ **M1**

(a) multiply by $2e^y$ and rearrange to $e^{2y} - 2x\,e^y - 1 = 0$ **M1**

(a) solve this quadratic in e^y using the quadratic formula: $e^y = x \pm \sqrt{x^2+1}$ **M1**

(a) reject the negative root since $e^y > 0$ but $x - \sqrt{x^2+1} < 0$ for all x **A1**

(a) hence $y = \ln(x + \sqrt{x^2+1})$, i.e. arsinh x = $\ln(x+\sqrt{x^2+1})$, cso **A1**

(a) Answer: arsinh x = $\ln(x + \sqrt{x^2+1})$

(b) substitute x = 2 into the formula from part (a) **M1**

(b) arsinh(2) = $\ln(2 + \sqrt{5})$, cao **A1**

(b) Answer: arsinh(2) = $\ln(2 + \sqrt{5})$

(c) take sinh of both sides: x = sinh(ln 3) = (3 - 1/3)/2 **M1**

(c) simplify (3 - 1/3)/2 = (8/3)/2 **M1**

(c) x = 4/3, cao **A1**

(c) Answer: x = 4/3

Question 18

(a) apply the chain rule to obtain the form k . x . $\cosh(3x^2-1)$ **M1**

(a) dy/dx = $6x \cosh(3x^2-1)$, cao **A1**

(a) Answer: dy/dx = $6x \cosh(3x^2 - 1)$

(b) use d/dx[ln u] = u'/u with u = cosh 2x, u' = 2 sinh 2x **M1**

(b) dy/dx = 2 sinh 2x / cosh 2x **A1**

(b) dy/dx = 2 tanh 2x, cao **A1**

(b) Answer: dy/dx = 2 tanh 2x

(c) write cosh y = 3x and differentiate implicitly: sinh y . dy/dx = 3 **M1**

(c) use sinh y = $\sqrt{\cosh^2(y) - 1}$ = $\sqrt{9x^2-1}$, taking the positive root since y > 0 **M1**

(c) dy/dx = $3/\sqrt{9x^2-1}$, cso **A1**

(c) Answer: dy/dx = $3/\sqrt{9x^2 - 1}$

(d) identify the point (0, 0) since arsinh(0) = 0 **B1**

(d) differentiate using d/dx[arsinh(u)] = $u'/\sqrt{u^2+1}$ with u = x/2, u' = 1/2 **M1**

(d) dy/dx at x = 0 is $(1/2)/\sqrt{0+1}$ = 1/2 **A1**

(d) tangent: y = (1/2)x, oe (e.g. 2y = x), cao **A1**

(d) Answer: y = (1/2)x

Question 19

(a) form k cosh(4x-1) **M1**

(a) (1/4)cosh(4x-1) + c, cao **A1**

(a) Answer: (1/4)cosh(4x-1) + c

(b) quote the standard result integral of $1/\sqrt{x^2+a^2}$ dx = arsinh(x/a) + c **B1**

(b) identify $a^2 = 9$, so a = 3 **M1**

(b) arsinh(x/3) + c, cao **A1**

(b) Answer: arsinh(x/3) + c

(c) substitute x = 4 sinh t, dx = 4 cosh t dt **M1**

(c) use $1 + \sinh^2(t) = \cosh^2(t)$ to write $\sqrt{x^2+16}$ = $\sqrt{16\sinh^2(t)+16}$ = 4cosh t **M1**

(c) integral becomes integral of (4cosh t)/(4cosh t) dt = integral of 1 dt **A1**

(c) = t + c, and t = arsinh(x/4) since x/4 = sinh t **A1**

(c) integral of $1/\sqrt{x^2+16}$ dx = arsinh(x/4) + c, cso (answer printed) **A1**

(c) Answer: arsinh(x/4) + c

(d) substitute limits: arsinh(1) - arsinh(0) **M1**

(d) = $\ln(1 + \sqrt{2})$, cao **A1**

(d) Answer: $\ln(1 + \sqrt{2})$

Question 20

(a) write cosh x cosh y = $(e^x+e^{-x})(e^y+e^{-y})/4$ and sinh x sinh y = $(e^x-e^{-x})(e^y-e^{-y})/4$ **M1**

(a) expand both products correctly in terms of e^{x+-y} **A1**

(a) add the two expansions, cancelling the e^{x-y} and e^{-x+y} terms with opposite sign **M1**

(a) result simplifies to $(e^{x+y}+e^{-(x+y)})/2$ = cosh(x+y), cso **A1**

(a) Answer: cosh(x+y) = cosh x cosh y + sinh x sinh y

(b) set y = x in the result from part (a): $\cosh 2x = \cosh^2(x) + \sinh^2(x)$ **M1**

(b) substitute $\sinh^2(x) = \cosh^2(x) - 1$ **M1**

(b) $\cosh 2x = 2\cosh^2(x) - 1$, cso (answer printed) **A1**

(b) Answer: $\cosh 2x = 2\cosh^2(x) - 1$

(c) substitute $\cosh 2x = 2\cosh^2(x) - 1$ into the equation **M1**

(c) rearrange to the quadratic $2\cosh^2(x) - 5\cosh x + 3 = 0$ **M1**

(c) factorise (2cosh x - 3)(cosh x - 1) = 0 to give cosh x = 3/2 or cosh x = 1 **M1**

(c) cosh x = 1 gives x = 0 **A1**

(c) cosh x = 3/2 gives x = +-ln((3+$\sqrt{5}$)/2) (awrt ±0.962), cao **A1**

(c) Answer: x = 0, x = ln((3+$\sqrt{5}$)/2), x = -ln((3+$\sqrt{5}$)/2) (awrt 0, 0.962, -0.962)

Question 21

(a) apply the standard result with a = 3: (1/3) artanh(x/3) + c **M1**

(a) substitute artanh(x/3) = (1/2) ln((1+x/3)/(1-x/3)) **M1**

(a) simplify (1/3)(1/2)ln((1+x/3)/(1-x/3)) to (1/6) ln((3+x)/(3-x)) **A1**

(a) integral of $1/(9-x^2)$ dx = (1/6) ln((3+x)/(3-x)) + c, cso (answer printed) **A1**

(a) Answer: (1/6) ln((3+x)/(3-x)) + c

(b) substitute limits: (1/6)ln(4/2) - (1/6)ln(3/3) **M1**

(b) simplify to (1/6) ln 2, since ln(3/3) = ln 1 = 0 **A1**

(b) k = 2, cao **A1**

(b) Answer: (1/6) ln 2, so k = 2

Question 22

(a) substitute $\sinh^2(x) = \cosh^2(x) - 1$ **M1**

(a) simplify to $3\cosh^2(x) - 5\cosh x = 0$, oe **A1**

(a) factorise to $\cosh x (3\cosh x - 5) = 0$, cso (answer printed) **A1**

(a) Answer: $\cosh x (3 \cosh x - 5) = 0$

(b) set each factor to zero: $\cosh x = 0$ or $\cosh x = 5/3$ **M1**

(b) reject $\cosh x = 0$, since $\cosh x \geq 1$ for all real x (no real solution) **B1**

(b) $\cosh x = 5/3$ gives $x = \pm\text{arcosh}(5/3)$ **M1**

(b) $\text{arcosh}(5/3) = \ln(5/3 + \sqrt{25/9-1}) = \ln(5/3+4/3) = \ln 3$ **A1**

(b) $x = \ln 3$ or $x = -\ln 3$, cao **A1**

(b) Answer: $x = \ln 3$ or $x = -\ln 3$