1.9 Proof by Induction: 1-hour Pack

EDEXCEL 9FM0 · Calculator allowed · about 60 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 A sequence is defined by $u_1 = 5$ and $u_{n+1} = 2u_n - 3$ for $n \geq 1$.

(a) Find the values of u_2 and u_3. **(2)**

(b) Prove by induction that $u_n = 2^n + 3$ for all positive integers n. **(5)**

(Total for Question 1 is 7 marks)

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2 Let $f(x) = x\,e^x$.

(a) Using the product rule, find f'(x), and verify that this agrees with the formula $f^n(x) = (x+n)e^x$ when n = 1. **(2)**

(b) Prove by induction that the nth derivative of f is $f^n(x) = (x+n)e^x$ for all positive integers n. **(4)**

(Total for Question 2 is 6 marks)

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3 Let $A(n) = 1^2 - 2^2 + 3^2 - 4^2 + \ldots + (-1)^{n+1}\,n^2$ (i.e. the sum from r=1 to n of $(-1)^{r+1}\,r^2$).

(a) Verify that the statement $A(n) = (-1)^{n+1}\,n(n+1)/2$ is true for n = 1 and for n = 2. **(1)**

(b) Prove by induction that $A(n) = (-1)^{n+1}\,n(n+1)/2$ for all positive integers n. **(5)**

(Total for Question 3 is 6 marks)

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4 Consider the statement W(n): $5^n + 9^n + 2$ is divisible by 4, for all positive integers n.

(a) Verify that W(1) is true. **(1)**

(b) Prove by induction that $5^n + 9^n + 2$ is divisible by 4 for all positive integers n. **(5)**

(Total for Question 4 is 6 marks)

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5 Prove by induction that $6^n - 1$ is divisible by 5 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

(Total for Question 5 is 5 marks)

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6 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $2r^3$ is equal to $2n^2(n + 1)^2/4$.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 6 is 5 marks)

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7 The matrix M is given by M = ((4, 0), (3, 1)). Prove by induction that, for all positive integers n, M^n = ((4^n, 0), ((4^n - 1), 1)), where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 7 is 5 marks)

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8 Prove by induction that $7^{2n} - 4^{2n}$ is divisible by 33 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

(Total for Question 8 is 5 marks)

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9 Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

(Total for Question 9 is 5 marks)

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Mark scheme · 1.9 Proof by Induction: 1-hour Pack

Question 1

(a) $u_2 = 2(5) - 3$ and $u_3 = 2u_2 - 3$ **M1**

(a) $u_2 = 7$, $u_3 = 11$ cao **A1**

(a) Answer: $u_2 = 7$, $u_3 = 11$

(b) basis n=1: $2^1+3 = 5 = u_1$, true **B1**

(b) assume true for n=k: $u_k = 2^k+3$ **M1**

(b) substitute into recurrence: $u_{k+1} = 2(2^k+3) - 3$ **dM1**

(b) simplify to $2^{k+1} + 3$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $u_{k+1} = 2^{k+1}+3$; true for all positive integers n by induction.

Question 2

(a) product rule: $f'(x) = e^x + x\,e^x$ **M1**

(a) $= (x+1)e^x$, which matches $(x+n)e^x$ with n=1 **A1**

(a) Answer: $f'(x) = (x+1)e^x$

(b) assume true for n=k: $f^k(x) = (x+k)e^x$ **M1**

(b) differentiate using the product rule: $f^{k+1}(x) = e^x + (x+k)e^x$ **M1**

(b) simplify to $(x+k+1)e^x$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $f^{k+1}(x) = (x+(k+1))e^x$; true for all positive integers n by induction.

Question 3

(a) n=1: LHS=1, RHS=1; n=2: LHS=1-4=-3, RHS=(-1)(3)=-3, both true **B1**

(a) Answer: Both cases hold: n=1 gives 1=1; n=2 gives -3=-3.

(b) assume true for n=k: $A(k) = (-1)^{k+1}\,k(k+1)/2$, form $A(k+1) = A(k) + (-1)^{k+2}(k+1)^2$ **M1**

(b) factor out $(-1)^{k+1}(k+1)$: $A(k+1) = (-1)^{k+1}(k+1)[\,k/2 - (k+1)\,]$ **M1**

(b) simplify the bracket to $-(k+2)/2$ and combine the sign correctly **dM1**

(b) simplify to $(-1)^{k+2}(k+1)(k+2)/2$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $A(k+1) = (-1)^{(k+1)+1}\,(k+1)(k+2)/2$; true for all positive integers n by induction.

Question 4

(a) 5+9+2 = 16 = 4(4), divisible by 4 **B1**

(a) Answer: n = 1: 5 + 9 + 2 = 16, divisible by 4.

(b) assume true for n=k: $5^k+9^k+2 = 4m$ for some positive integer m **M1**

(b) consider $5^{k+1}+9^{k+1}+2 - 5(5^k+9^k+2)$ and expand **M1**

(b) simplify this difference to $4(9^k - 2)$ **dM1**

(b) hence $5^{k+1}+9^{k+1}+2 = 5(4m) + 4(9^k-2) = 4(5m+9^k-2)$, divisible by 4, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $5^{k+1}+9^{k+1}+2 = 4(5m+9^k-2)$; true for all positive integers n by induction.

Question 5

(a) f(1) = 5 x 1, divisible by 5 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 6^{k+1} - 1 = 6(6^k - 1) + 5 = 6f(k) + 5$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 6

(a) LHS = 2, RHS = 2, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 2

(b) assumes sum to $k = 2k^2(k + 1)^2/4$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $2(k + 1)^2((k + 1) + 1)^2/4$ by the simplification: $2k^2(k + 1)^2/4 + 2(k + 1)^3 = 2(k + 1)^2[k^2 + 4(k + 1)]/4 = 2(k + 1)^2(k + 2)^2/4$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 7

(a) evaluates the formula at n = 1: ((4, 0), (3, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: $[(4^k - 1)] \times 4 + 3 = (4^{k+1} - 1)$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

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Question 8

(a) f(1) = 49 - 16 = 33, divisible by 33 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k + 2} - 4^{2k + 2} = 49f(k) + 4^{2k}(49 - 16) = 49f(k) + 33 \times 4^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 9

(a) f(1) = 7 x 1, divisible by 7 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 8^{k + 1} - 1 = 8(8^{k} - 1) + 7 = 8f(k) + 7$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

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