# 1.9 Proof by Induction: 2-hour Pack

EDEXCEL 9FM0 · Calculator allowed · about 120 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

**Answer ALL questions. Show all your working.**

**1** The sequence $u_n$ is defined by $u_1 = 11$, $u_2 = 43$ and $u_{n+2} = 8u_{n+1} - 15u_n$ for $n \geq 1$. Prove by induction that $u_n = 2 \times 3^n + 1 \times 5^n$ for all positive integers n.

**(a)** Show that the formula is true for n = 1 and n = 2. **(2)**

**(b)** Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

**(Total for Question 1 is 6 marks)**

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**2** The sequence $u_n$ is defined by $u_1 = -4$, $u_2 = -32$ and $u_{n+2} = 8u_{n+1} - 15u_n$ for $n \geq 1$. Prove by induction that $u_n = 2 \times 3^n - 2 \times 5^n$ for all positive integers n.

**(a)** Show that the formula is true for n = 1 and n = 2. **(2)**

**(b)** Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

**(Total for Question 2 is 6 marks)**

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**3** Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $r^3$ is equal to $n^2(n + 1)^2/4$.

**(a)** State the base case n = 1 and verify the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

**(Total for Question 3 is 5 marks)**

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**4** Prove by induction that $7^{2n} - 5^{2n}$ is divisible by 24 for all positive integers n.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

**(Total for Question 4 is 5 marks)**


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**5** The matrix M is given by M = ((2, 1), (0, 2)). Prove by induction that, for all positive integers n, $M^n = ((2^n, n \times 2^{n-1}), (0, 2^n))$, where each matrix is written row by row.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

**(Total for Question 5 is 5 marks)**

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**6** Prove by induction that $7^{2n} - 6^{2n}$ is divisible by 13 for all positive integers n.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

**(Total for Question 6 is 5 marks)**

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**7** The sequence $u_n$ is defined by $u_1 = 7$, $u_2 = 29$ and $u_{n+2} = 7u_{n+1} - 10u_n$ for $n \geq 1$. Prove by induction that $u_n = 1 \times 2^n + 1 \times 5^n$ for all positive integers n.

**(a)** Show that the formula is true for n = 1 and n = 2. **(2)**

**(b)** Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

**(Total for Question 7 is 6 marks)**

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**8** The matrix M is given by M = ((1, 2), (0, 1)). Prove by induction that, for all positive integers n, $M^n = ((1, 2n), (0, 1))$, where each matrix is written row by row.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

**(Total for Question 8 is 5 marks)**

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**9** The sequence $u_n$ is defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3$. Prove by induction that $u_n = 6 \times 2^{n-1} - 3$ for all positive integers n.

**(a)** Show that the formula gives the correct value for n = 1. **(1)**

**(b)** Assume true for n = k and prove it is true for n = k + 1, and state the conclusion. **(4)**

**(Total for Question 9 is 5 marks)**


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**10** Prove by induction that $3^n - 1$ is divisible by 2 for all positive integers n.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

**(Total for Question 10 is 5 marks)**

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**11** Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $r \times 2^r$ is equal to $(n - 1)2^{n + 1} + 2$.

**(a)** State the base case n = 1 and verify the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

**(Total for Question 11 is 5 marks)**

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**12** The sequence $u_n$ is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 6$. Prove by induction that $u_n = 5 \times 3^{n - 1} - 3$ for all positive integers n.

**(a)** Show that the formula gives the correct value for n = 1. **(1)**

**(b)** Assume true for n = k and prove it is true for n = k + 1, and state the conclusion. **(4)**

**(Total for Question 12 is 5 marks)**

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**13** Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $3/(r(r + 1))$ is equal to $3n/(n + 1)$.

**(a)** State the base case n = 1 and verify the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

**(Total for Question 13 is 5 marks)**

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**14** The matrix M is given by M = ((1, 5), (0, 1)). Prove by induction that, for all positive integers n, $M^n$ = ((1, 5n), (0, 1)), where each matrix is written row by row.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

**(Total for Question 14 is 5 marks)**


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**15** The matrix M is given by M = ((3, 0), (4, 1)). Prove by induction that, for all positive integers n, $M^n = ((3^n, 0), (2(3^n - 1), 1))$, where each matrix is written row by row.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

**(Total for Question 15 is 5 marks)**

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**16** Prove by induction that $5^{2n} - 4^{2n}$ is divisible by 9 for all positive integers n.

**(a)** Show that the result is true for n = 1. **(1)**

**(b)** Assume the result is true for n = k, where f(k) is divisible by the stated number. Show that it is then true for n = k + 1 and conclude. **(4)**

**(Total for Question 16 is 5 marks)**

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**17** The matrix M is given by M = ((3, 0), (1, 1)), where rows are written left to right, top row then bottom row.

**(a)** By direct matrix multiplication, find $M^2$, and verify that it agrees with the formula $M^n = ((3^n, 0), ((3^n - 1)/2, 1))$ when n = 2. **(2)**

**(b)** Prove by induction that $M^n = ((3^n, 0), ((3^n - 1)/2, 1))$ for all positive integers n. **(5)**

**(Total for Question 17 is 7 marks)**

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**18** You may use the addition formulae cos(A+B) = cosA cosB - sinA sinB and sin(A+B) = sinA cosB + cosA sinB.

**(a)** Verify that the statement $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ is true for n = 1. **(1)**

**(b)** Prove by induction that $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ for all positive integers n. **(5)**

**(c)** State the value predicted by the formula for $(\cos(\theta) + i \sin(\theta))^0$, and confirm this is consistent with the usual convention for any non-zero complex number raised to the power 0. **(1)**

**(Total for Question 18 is 7 marks)**


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# Mark scheme · 1.9 Proof by Induction: 2-hour Pack

## Question 1

(a) n = 1: 2(3) + 1(5) = 11 **B1**

(a) n = 2: 2(9) + 1(25) = 43 **B1**

**(a) Answer: true for n = 1 and n = 2**

(b) $u_{k+2} = 8u_{k+1} - 15u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $2 \times 3^k(8 \times 3 - 15) = 2 \times 3^{k+2}$ since $3^2 - 8 \times 3 + 15 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $2 \times 3^{k+2} + 1 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

**(b) Answer: proved by induction**

## Question 2

(a) n = 1: 2(3) - 2(5) = -4 **B1**

(a) n = 2: 2(9) - 2(25) = -32 **B1**

**(a) Answer: true for n = 1 and n = 2**

(b) $u_{k+2} = 8u_{k+1} - 15u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $2 \times 3^k(8 \times 3 - 15) = 2 \times 3^{k+2}$ since $3^2 - 8 \times 3 + 15 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $2 \times 3^{k+2} - 2 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

**(b) Answer: proved by induction**

## Question 3

(a) LHS = 1, RHS = 1, so true for n = 1 **B1**

**(a) Answer: n = 1: both sides equal 1**

(b) assumes sum to k = $k^2(k + 1)^2/4$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $(k + 1)^2((k + 1) + 1)^2/4$ by the simplification: $k^2(k + 1)^2/4 + 1(k + 1)^3 = 1(k + 1)^2[k^2 + 4(k + 1)]/4 = 1(k + 1)^2(k + 2)^2/4$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 4

(a) f(1) = 49 - 25 = 24, divisible by 24 **B1**

**(a) Answer: true for n = 1**

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k+2} - 5^{2k+2} = 49f(k) + 5^{2k}(49 - 25) = 49f(k) + 24 \times 5^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 5

(a) evaluates the formula at n = 1: ((2, 1), (0, 2)) = M **B1**

**(a) Answer: true for n = 1**

(b) assumes $M^k$ as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: $2^k \times 1 + k2^{k-1} \times 2 = (k + 1)2^k$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 6

(a) f(1) = 49 - 36 = 13, divisible by 13 **B1**

**(a) Answer: true for n = 1**

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k+2} - 6^{2k+2} = 49f(k) + 6^{2k}(49 - 36) = 49f(k) + 13 \times 6^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 7

(a) n = 1: 1(2) + 1(5) = 7 **B1**

(a) n = 2: 1(4) + 1(25) = 29 **B1**

**(a) Answer: true for n = 1 and n = 2**

(b) $u_{k+2} = 7u_{k+1} - 10u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $1 \times 2^k(7 \times 2 - 10) = 1 \times 2^{k+2}$ since $2^2 - 7 \times 2 + 10 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $1 \times 2^{k+2} + 1 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

**(b) Answer: proved by induction**

## Question 8

(a) evaluates the formula at n = 1: ((1, 2), (0, 1)) = M **B1**

**(a) Answer: true for n = 1**

(b) assumes $M^k$ as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: 1 x 2 + 2k x 1 = 2(k + 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**


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## Question 9

(a) $u_1 = 6 \times 2^0 - 3 = 3$ **B1**

**(a) Answer: true for n = 1**

(b) $u_{k+1} = 2u_k + 3 = 2(6 \times 2^{k-1} - 3) + 3$ **M1**

(b) expands: $6 \times 2^k - 6 + 3$ **M1**

(b) $3 - 6 = -3$, giving $6 \times 2^k - 3$ **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all n by induction (cso) **A1**

**(b) Answer: proved by induction**

## Question 10

(a) f(1) = 2 x 1, divisible by 2 **B1**

**(a) Answer: true for n = 1**

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k+1) = 3^{k+1} - 1 = 3(3^k - 1) + 2 = 3f(k) + 2$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 11

(a) LHS = 2, RHS = 2, so true for n = 1 **B1**

**(a) Answer: n = 1: both sides equal 2**

(b) assumes sum to k = $(k-1)2^{k+1} + 2$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $((k+1)-1)2^{(k+1)+1} + 2$ by the simplification: $(k-1)2^{k+1} + 2 + (k+1)2^{k+1} = 2^{k+1}(2k) + 2 = k\,2^{k+2} + 2$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 12

(a) $u_1 = 5 \times 3^0 - 3 = 2$ **B1**

**(a) Answer: true for n = 1**

(b) $u_{k+1} = 3u_k + 6 = 3(5 \times 3^{k-1} - 3) + 6$ **M1**

(b) expands: $5 \times 3^k - 9 + 6$ **M1**

(b) $6 - 9 = -3$, giving $5 \times 3^k - 3$ **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all n by induction (cso) **A1**

**(b) Answer: proved by induction**

## Question 13

(a) LHS = 3/2, RHS = 3/2, so true for n = 1 **B1**

**(a) Answer: n = 1: both sides equal 3/2**

(b) assumes sum to k = 3k/(k + 1) and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $3(k+1)/((k+1)+1)$ by the simplification: $3k/(k+1) + 3/((k+1)(k+2)) = 3[k(k+2)+1]/((k+1)(k+2)) = 3(k+1)/(k+2)$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 14

(a) evaluates the formula at n = 1: ((1, 5), (0, 1)) = M **B1**

**(a) Answer: true for n = 1**

(b) assumes $M^k$ as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: 1 x 5 + 5k x 1 = 5(k + 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 15

(a) evaluates the formula at n = 1: ((3, 0), (4, 1)) = M **B1**

**(a) Answer: true for n = 1**

(b) assumes $M^k$ as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: $[2(3^k - 1)] \times 3 + 4 = 2(3^{k+1} - 1)$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 16

(a) f(1) = 25 - 16 = 9, divisible by 9 **B1**

**(a) Answer: true for n = 1**

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k+1) = 5^{2k+2} - 4^{2k+2} = 25f(k) + 4^{2k}(25 - 16) = 25f(k) + 9 \times 4^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

**(b) Answer: result proved by induction**

## Question 17

(a) multiply M by M to get $M^2 = ((9, 0), (4, 1))$ **M1**

(a) $(3^2-1)/2 = 4$, matching the bottom-left entry **A1**

**(a) Answer: $M^2 = ((9, 0), (4, 1))$**

(b) assume true for n=k: $M^k = ((3^k,0),((3^k-1)/2,1))$, form $M^{k+1} = M^k * M$ **M1**

(b) carry out the matrix multiplication $M^k * M$ correctly **M1**

(b) simplify the bottom-left entry: $3(3^k-1)/2 + 1$ **dM1**

(b) simplify to $(3^{k+1}-1)/2$, and top-left to $3^{k+1}$, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

**(b) Answer: $M^{k+1} = ((3^{k+1},0),((3^{k+1}-1)/2,1))$; true for all positive integers n by induction.**


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## Question 18

(a) for n=1, LHS = cos(θ)+isin(θ) = RHS = cos(1.θ)+isin(1.θ), so true **B1**

**(a) Answer: n = 1: both sides equal cos(θ) + i sin(θ).**

(b) assume true for n=k: $(\cos\theta + i\sin\theta)^k = \cos(k\theta)+i\sin(k\theta)$ **M1**

(b) write $(\cos\theta+i\sin\theta)^{k+1} = (\cos(k\theta)+i\sin(k\theta))(\cos\theta+i\sin\theta)$ and expand **M1**

(b) collect real and imaginary parts using the addition formulae **dM1**

(b) simplify to cos((k+1)θ) + isin((k+1)θ) cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

**(b) Answer: $(\cos\theta+i\sin\theta)^{k+1} = \cos((k+1)\theta)+i\sin((k+1)\theta)$; true for all positive integers n by induction (De Moivre's theorem).**

(c) formula gives cos(0)+isin(0) = 1, consistent with $z^0 = 1$ for any non-zero z **B1**

**(c) Answer: cos(0) + i sin(0) = 1, consistent with $z^0 = 1$.**


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