1.9 Proof by Induction: Test Pack

EDEXCEL 9FM0 · Calculator allowed · about 60 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 Let $S(n) = 1^2 + 2^2 + 3^2 + ... + n^2$.

(a) Verify that the statement $S(n) = (1/6)n(n+1)(2n+1)$ is true for $n = 1$. **(1)**

(b) Assuming the statement is true for $n = k$, prove that it is then true for $n = k + 1$, and hence prove by induction that $S(n) = (1/6)n(n+1)(2n+1)$ for all positive integers n. **(4)**

(c) Hence find the value of $1^2 + 2^2 + ... + 20^2$. **(2)**

(Total for Question 1 is 7 marks)

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2 Consider the statement Q(n): $n^3 + 5n$ is divisible by 6, for all positive integers n.

(a) Verify that Q(1) is true. **(1)**

(b) Prove by induction that $n^3 + 5n$ is divisible by 6 for all positive integers n. You should explain clearly why any additional term you introduce is divisible by 6. **(5)**

(c) Hence explain briefly why $n^3 + 5n$ must also be an even number for every positive integer n. **(1)**

(Total for Question 2 is 7 marks)

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3 Consider the statement R(n): $n! > 3^n$, for integers $n \geq 7$.

(a) Show that R(6) is false and that R(7) is true, and explain why n = 7 is the correct starting value for an induction proof of R(n). **(2)**

(b) Prove by induction that $n! > 3^n$ for all integers $n \geq 7$. **(5)**

(Total for Question 3 is 7 marks)

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4 The matrix M is given by M = ((2, 0), (5, 1)). Prove by induction that, for all positive integers n, $M^n = ((2^n, 0), (5(2^n - 1), 1))$, where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 4 is 5 marks)

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5 The matrix M is given by M = ((2, 0), (1, 1)). Prove by induction that, for all positive integers n, $M^n = ((2^n, 0), ((2^n - 1), 1))$, where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 5 is 5 marks)

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6 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of r x r! is equal to (n + 1)! - 1.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 6 is 5 marks)

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7 A sequence is defined by $a_1 = 1$, $a_2 = 1$, and $a_{n+2} = a_{n+1} + a_n$ for $n \geq 1$ (the Fibonacci sequence).

(a) Find a_3 and a_4, and verify that $a_n < 2^n$ for n = 1, 2, 3 and 4. **(2)**

(b) Explain why, in order to prove $a_{k+1} < 2^{k+1}$ using the recurrence relation, it is necessary to assume the inequality is true for both n = k and n = k - 1, rather than for n = k alone. **(1)**

(c) Using this two-step approach, prove by induction that $a_n < 2^n$ for all positive integers n. **(6)**

(Total for Question 7 is 9 marks)

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Mark scheme · 1.9 Proof by Induction: Test Pack

Question 1

(a) LHS = 1, RHS = (1/6)(1)(2)(3) = 1, so true for n=1 **B1**

(a) Answer: n = 1: LHS = 1, RHS = 1, statement holds.

(b) assume true for n=k: S(k) = (1/6)k(k+1)(2k+1), then form S(k+1) = S(k) + $(k+1)^2$ **M1**

(b) factorise out (k+1): S(k+1) = (k+1)[(1/6)k(2k+1) + (k+1)], combine over common denominator 6 **dM1**

(b) correct simplification to (1/6)(k+1)(k+2)(2k+3) cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction (dep on all previous marks) **A1**

(b) Answer: S(k+1) = (1/6)(k+1)(k+2)(2(k+1)+1); true for all positive integers n by induction.

(c) substitute n=20 into (1/6)n(n+1)(2n+1) **M1**

(c) 2870 cao **A1**

(c) Answer: 2870

Question 2

(a) 1^3 + 5(1) = 6, which is divisible by 6 **B1**

(a) Answer: n = 1: 1 + 5 = 6, divisible by 6.

(b) assume true for n=k: k^3 + 5k = 6m for some positive integer m, form $(k+1)^3$ + 5(k+1) **M1**

(b) expand and group as (k^3+5k) + $3k^2$ + 3k + 6 = 6m + 3k(k+1) + 6 **M1**

(b) explain that k(k+1) is a product of two consecutive integers so is always even, hence 3k(k+1) is divisible by 6 **B1**

(b) conclude $(k+1)^3$+5(k+1) = 6(m + j + 1) for k(k+1)=2j, divisible by 6, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $(k+1)^3$+5(k+1) = 6(m+j+1) where k(k+1)=2j; true for all positive integers n by induction.

(c) since n^3+5n is divisible by 6 = 2x3, it must be divisible by 2, so it is always even **B1**

(c) Answer: Any multiple of 6 is also a multiple of 2, so n^3+5n is always even.

Question 3

(a) n=6: 6≠720, 3^6=729, so 720<729 and R(6) is false **B1**

(a) n=7: 7≠5040, 3^7=2187, so 5040>2187 and R(7) is true; n=7 is the smallest value for which the inequality holds, so it is the valid base case **B1**

(a) Answer: R(6) false (720 < 729); R(7) true (5040 > 2187); n=7 is the smallest valid base case.

(b) assume true for n=k (k≥7): k! > 3^k, form (k+1)! = (k+1)*k! **M1**

(b) use the assumption: (k+1)! > (k+1)*3^k **M1**

(b) argue that since k≥7, k+1≥8>3, so (k+1)*3^k > 3*3^k = 3^{k+1} **dM1**

(b) combine to give (k+1)! > 3^{k+1} cso **A1**

(b) conclusion: true for n=7, true for n=k implies true for n=k+1, so true for all integers n ≥ 7 by mathematical induction **A1**

(b) Answer: (k+1)! > 3^{k+1}; true for all integers n ≥ 7 by induction.

Question 4

(a) evaluates the formula at n = 1: ((2, 0), (5, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes M^{k+1} = M^k M **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: [5(2^k - 1)] x 2 + 5 = 5(2^{k+1} - 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 5

(a) evaluates the formula at n = 1: ((2, 0), (1, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes M^{k+1} = M^k M **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: [(2^k - 1)] x 2 + 1 = (2^{k+1} - 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 6

(a) LHS = 1, RHS = 1, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 1

(b) assumes sum to k = (k + 1)! - 1 and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains ((k + 1) + 1)! - 1 by the simplification: (k + 1)! - 1 + (k + 1)(k + 1)! = (k + 1)!(1 + k + 1) - 1 = (k + 2)(k + 1)! - 1 = (k + 2)! - 1 **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

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Question 7

(a) $a_3 = a_2+a_1 = 2$, $a_4 = a_3+a_2 = 3$ **M1**

(a) check each: $1<2$, $1<4$, $2<8$, $3<16$, all true cao **A1**

(a) Answer: $a_3 = 2$, $a_4 = 3$; inequality holds for n = 1, 2, 3, 4.

(b) since $a_{k+1} = a_k + a_{k-1}$, the recurrence involves two previous terms, so both $a_k < 2^k$ and $a_{k-1} < 2^{k-1}$ are needed to bound a_{k+1}; this requires two base cases and a two-step (strong) inductive hypothesis **B1**

(b) Answer: The recurrence $a_{k+1} = a_k + a_{k-1}$ depends on two previous terms, so the inductive hypothesis must cover both n = k and n = k-1 (strong induction with two base cases).

(c) basis n=1: $a_1 = 1 < 2^1 = 2$ **B1**

(c) basis n=2: $a_2 = 1 < 2^2 = 4$ **B1**

(c) assume true for n=k-1 and n=k ($k \geq 2$): $a_{k-1}<2^{k-1}$ and $a_k<2^k$ **M1**

(c) use the recurrence: $a_{k+1} = a_k + a_{k-1} < 2^k + 2^{k-1}$ **M1**

(c) since $2^{k-1}<2^k$, $2^k+2^{k-1} < 2^k+2^k = 2^{k+1}$, so $a_{k+1} < 2^{k+1}$ cso **A1**

(c) conclusion: true for n=1 and n=2, and true for n=k-1,k implies true for n=k+1, so true for all positive integers n by (strong) mathematical induction **A1**

(c) Answer: $a_{k+1} < 2^{k+1}$; true for all positive integers n by strong induction.

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