1.9 Proof by Induction

EDEXCEL 9FM0 · Calculator allowed · about 230 minutes · Write-in edition

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 Let $T(n) = 1^3 + 2^3 + 3^3 + ... + n^3$.

(a) Verify that the statement $T(n) = [n(n+1)/2]^2$ is true for $n = 1$. **(1)**

(b) Prove by induction that $T(n) = [n(n+1)/2]^2$ for all positive integers n. **(4)**

(c) Show that $1^3 + 2^3 + ... + 10^3 = 3025$, and verify that this equals $(1 + 2 + ... + 10)^2$. **(2)**

(Total for Question 1 is 7 marks)

2 Let U(n) = 1(2) + 2(3) + 3(4) + ... + n(n+1).

(a) Verify that the statement U(n) = (1/3)n(n+1)(n+2) is true for n = 1. **(1)**

(b) Prove by induction that U(n) = (1/3)n(n+1)(n+2) for all positive integers n. **(4)**

(c) Hence find the value of 1(2) + 2(3) + ... + 15(16). **(2)**

(Total for Question 2 is 7 marks)

3 Consider the statement P(n): $3^{2n} - 1$ is divisible by 8, for all positive integers n.

(a) Verify that P(1) is true. **(1)**

(b) Prove by induction that $3^{2n} - 1$ is divisible by 8 for all positive integers n. **(5)**

(c) By testing the case n = 2, determine whether the statement '$3^{2n} - 1$ is divisible by 24 for all positive integers n' is true or false. **(2)**

(Total for Question 3 is 8 marks)

4 A sequence is defined by $u_1 = 5$ and $u_{n+1} = 2u_n - 3$ for $n \geq 1$.

(a) Find the values of u_2 and u_3. **(2)**

(b) Prove by induction that $u_n = 2^n + 3$ for all positive integers n. **(5)**

(Total for Question 4 is 7 marks)

5 Let $f(x) = x\,e^x$.

(a) Using the product rule, find $f'(x)$, and verify that this agrees with the formula $f^n(x) = (x+n)e^x$ when $n = 1$. **(2)**

(b) Prove by induction that the nth derivative of f is $f^n(x) = (x+n)e^x$ for all positive integers n. **(4)**

(Total for Question 5 is 6 marks)

6 Let $A(n) = 1^2 - 2^2 + 3^2 - 4^2 + ... + (-1)^{n+1} n^2$ (i.e. the sum from r=1 to n of $(-1)^{r+1} r^2$).

(a) Verify that the statement $A(n) = (-1)^{n+1} n(n+1)/2$ is true for n = 1 and for n = 2. **(1)**

(b) Prove by induction that $A(n) = (-1)^{n+1} n(n+1)/2$ for all positive integers n. **(5)**

(Total for Question 6 is 6 marks)

7 Consider the statement W(n): $5^n + 9^n + 2$ is divisible by 4, for all positive integers n.

(a) Verify that W(1) is true. **(1)**

(b) Prove by induction that $5^n + 9^n + 2$ is divisible by 4 for all positive integers n. **(5)**

(Total for Question 7 is 6 marks)

8 The sequence u_n is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 2$. Prove by induction that $u_n = 3 \times 3^{n-1} - 1$ for all positive integers n.

(a) Show that the formula gives the correct value for n = 1. **(1)**

(b) Assume true for n = k and prove it is true for n = k + 1, and state the conclusion. **(4)**

(Total for Question 8 is 5 marks)

9 Prove by induction that $6^n - 1$ is divisible by 5 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 9 is 5 marks)

10 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of r(r + 5) is equal to n(n + 1)(2n + 16)/6.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 10 is 5 marks)

11 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $2r^3$ is equal to $2n^2(n + 1)^2/4$.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 11 is 5 marks)

12 The matrix M is given by M = ((4, 0), (3, 1)). Prove by induction that, for all positive integers n, $M^n = ((4^n, 0), ((4^n - 1), 1))$, where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 12 is 5 marks)

13 Prove by induction that $7^{2n} - 4^{2n}$ is divisible by 33 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 13 is 5 marks)

14 Prove by induction that $8^n - 1$ is divisible by 7 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 14 is 5 marks)

15 The sequence u_n is defined by $u_1 = 11$, $u_2 = 43$ and $u_{n+2} = 8u_{n+1} - 15u_n$ for $n \geq 1$. Prove by induction that $u_n = 2 \times 3^n + 1 \times 5^n$ for all positive integers n.

(a) Show that the formula is true for n = 1 and n = 2. **(2)**

(b) Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

(Total for Question 15 is 6 marks)

16 The sequence u_n is defined by $u_1 = -4$, $u_2 = -32$ and $u_{n+2} = 8u_{n+1} - 15u_n$ for $n \geq 1$. Prove by induction that $u_n = 2 \times 3^n - 2 \times 5^n$ for all positive integers n.

(a) Show that the formula is true for n = 1 and n = 2. **(2)**

(b) Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

(Total for Question 16 is 6 marks)

17 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of r^3 is equal to $n^2(n + 1)^2/4$.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 17 is 5 marks)

18 Prove by induction that $7^{2n} - 5^{2n}$ is divisible by 24 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 18 is 5 marks)

19 The matrix M is given by M = ((2, 1), (0, 2)). Prove by induction that, for all positive integers n, $M^n = ((2^n, n \times 2^{n-1}), (0, 2^n))$, where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 19 is 5 marks)

20 Prove by induction that $7^{2n} - 6^{2n}$ is divisible by 13 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 20 is 5 marks)

21 The sequence u_n is defined by $u_1 = 7$, $u_2 = 29$ and $u_{n+2} = 7u_{n+1} - 10u_n$ for $n \geq 1$. Prove by induction that $u_n = 1 \times 2^n + 1 \times 5^n$ for all positive integers n.

(a) Show that the formula is true for n = 1 and n = 2. **(2)**

(b) Assume the formula is true for n = k and n = k + 1. Prove it is true for n = k + 2 and conclude. **(4)**

(Total for Question 21 is 6 marks)

22 The matrix M is given by M = ((1, 2), (0, 1)). Prove by induction that, for all positive integers n, M^n = ((1, 2n), (0, 1)), where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 22 is 5 marks)

23 The sequence u_n is defined by $u_1 = 3$ and $u_{n+1} = 2u_n + 3$. Prove by induction that $u_n = 6 \times 2^{n-1} - 3$ for all positive integers n.

(a) Show that the formula gives the correct value for n = 1. **(1)**

(b) Assume true for n = k and prove it is true for n = k + 1, and state the conclusion. **(4)**

(Total for Question 23 is 5 marks)

24 Prove by induction that $3^n - 1$ is divisible by 2 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 24 is 5 marks)

25 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of r x 2^r is equal to $(n - 1)2^{n+1} + 2$.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 25 is 5 marks)

26 The sequence u_n is defined by $u_1 = 2$ and $u_{n+1} = 3u_n + 6$. Prove by induction that $u_n = 5 \times 3^{n-1} - 3$ for all positive integers n.

(a) Show that the formula gives the correct value for n = 1. **(1)**

(b) Assume true for n = k and prove it is true for n = k + 1, and state the conclusion. **(4)**

(Total for Question 26 is 5 marks)

27 Prove by mathematical induction that, for all positive integers n, the sum from r = 1 to n of $3/(r(r + 1))$ is equal to $3n/(n + 1)$.

(a) State the base case n = 1 and verify the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, and prove it is then true for n = k + 1. Hence state the conclusion. **(4)**

(Total for Question 27 is 5 marks)

28 The matrix M is given by M = ((1, 5), (0, 1)). Prove by induction that, for all positive integers n, M^n = ((1, 5n), (0, 1)), where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 28 is 5 marks)

29 The matrix M is given by M = ((3, 0), (4, 1)). Prove by induction that, for all positive integers n, $M^n = ((3^n, 0), (2(3^n - 1), 1))$, where each matrix is written row by row.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k and prove it is true for n = k + 1, stating your conclusion. **(4)**

(Total for Question 29 is 5 marks)

30 Prove by induction that $5^{2n} - 4^{2n}$ is divisible by 9 for all positive integers n.

(a) Show that the result is true for n = 1. **(1)**

(b) Assume the result is true for n = k, where f(k) is divisible by the stated number. **(4)**
Show that it is then true for n = k + 1 and conclude.

(Total for Question 30 is 5 marks)

31 The matrix M is given by M = ((3, 0), (1, 1)), where rows are written left to right, top row then bottom row.

(a) By direct matrix multiplication, find M^2, and verify that it agrees with the formula $M^n = ((3^n, 0), ((3^n - 1)/2, 1))$ when n = 2. **(2)**

(b) Prove by induction that $M^n = ((3^n, 0), ((3^n - 1)/2, 1))$ for all positive integers n. **(5)**

(Total for Question 31 is 7 marks)

32 You may use the addition formulae cos(A+B) = cosA cosB - sinA sinB and sin(A+B) = sinA cosB + cosA sinB.

(a) Verify that the statement $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ is true for n = 1. **(1)**

(b) Prove by induction that $(\cos(\theta) + i \sin(\theta))^n = \cos(n\theta) + i \sin(n\theta)$ for all positive integers n. **(5)**

(c) State the value predicted by the formula for $(\cos(\theta) + i \sin(\theta))^0$, and confirm this is consistent with the usual convention for any non-zero complex number raised to the power 0. **(1)**

(Total for Question 32 is 7 marks)

Mark scheme · 1.9 Proof by Induction

Question 1

(a) LHS = 1, RHS = $[1*2/2]^2$ = 1, so true for n=1 **B1**

(a) Answer: n = 1: LHS = 1, RHS = 1, statement holds.

(b) assume true for n=k: T(k) = $[k(k+1)/2]^2$, form T(k+1) = T(k) + $(k+1)^3$ **M1**

(b) take out common factor $(k+1)^2$: T(k+1) = $(k+1)^2[\ k^2/4$ + (k+1)] **dM1**

(b) correct simplification to $[(k+1)(k+2)/2]^2$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: T(k+1) = $[(k+1)(k+2)/2]^2$; true for all positive integers n by induction.

(c) $[10(11)/2]^2 = 55^2$ = 3025 cso **B1**

(c) 1+2+...+10 = 55, and 55^2 = 3025, so the two are equal **B1**

(c) Answer: 3025; equals $(55)^2$, confirming the identity.

Question 2

(a) LHS = 1(2) = 2, RHS = (1/3)(1)(2)(3) = 2, so true for n=1 **B1**

(a) Answer: n = 1: LHS = 2, RHS = 2, statement holds.

(b) assume true for n=k: U(k) = (1/3)k(k+1)(k+2), form U(k+1) = U(k) + (k+1)(k+2) **M1**

(b) take out common factor (k+1)(k+2): U(k+1) = (k+1)(k+2)[k/3 + 1] **dM1**

(b) correct simplification to (1/3)(k+1)(k+2)(k+3) cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: U(k+1) = (1/3)(k+1)(k+2)(k+3); true for all positive integers n by induction.

(c) substitute n=15 into (1/3)n(n+1)(n+2) **M1**

(c) 1360 cao **A1**

(c) Answer: 1360

Question 3

(a) $3^2 - 1 = 8 = 8(1)$, which is divisible by 8 **B1**

(a) Answer: n = 1: $3^2 - 1 = 8$, divisible by 8.

(b) assume true for n=k: $3^{2k} - 1 = 8m$ for some positive integer m, so $3^{2k} = 8m + 1$ **M1**

(b) write $3^{2(k+1)} - 1 = 9(3^{2k}) - 1$ **M1**

(b) substitute $3^{2k} = 8m+1$ to get 9(8m+1) - 1 **dM1**

(b) simplify to 72m + 8 = 8(9m+1), which is divisible by 8, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $3^{2(k+1)} - 1 = 8(9m+1)$; true for all positive integers n by induction.

(c) n=2: $3^4 - 1 = 80$ **B1**

(c) 80 / 24 is not an integer (80 = 24(3) + 8), so the statement is false **B1**

(c) Answer: False: $3^4 - 1 = 80$, which is not divisible by 24.

Question 4

(a) $u_2 = 2(5) - 3$ and $u_3 = 2u_2 - 3$ **M1**

(a) $u_2 = 7$, $u_3 = 11$ cao **A1**

(a) Answer: $u_2 = 7$, $u_3 = 11$

(b) basis n=1: $2^1+3 = 5 = u_1$, true **B1**

(b) assume true for n=k: $u_k = 2^k+3$ **M1**

(b) substitute into recurrence: $u_{k+1} = 2(2^k+3) - 3$ **dM1**

(b) simplify to $2^{k+1} + 3$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $u_{k+1} = 2^{k+1}+3$; true for all positive integers n by induction.

Question 5

(a) product rule: $f'(x) = e^x + x\ e^x$ **M1**

(a) = $(x+1)e^x$, which matches $(x+n)e^x$ with n=1 **A1**

(a) Answer: $f'(x) = (x+1)e^x$

(b) assume true for n=k: $f^k(x) = (x+k)e^x$ **M1**

(b) differentiate using the product rule: $f^{k+1}(x) = e^x + (x+k)e^x$ **M1**

(b) simplify to $(x+k+1)e^x$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $f^{k+1}(x) = (x+(k+1))e^x$; true for all positive integers n by induction.

Question 6

(a) n=1: LHS=1, RHS=1; n=2: LHS=1-4=-3, RHS=(-1)(3)=-3, both true **B1**

(a) Answer: Both cases hold: n=1 gives 1=1; n=2 gives -3=-3.

(b) assume true for n=k: $A(k) = (-1)^{k+1}\ k(k+1)/2$, form $A(k+1) = A(k) + (-1)^{k+2}(k+1)^2$ **M1**

(b) factor out $(-1)^{k+1}(k+1)$: $A(k+1) = (-1)^{k+1}(k+1)[\ k/2$ - (k+1)] **M1**

(b) simplify the bracket to -(k+2)/2 and combine the sign correctly **dM1**

(b) simplify to $(-1)^{k+2}(k+1)(k+2)/2$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $A(k+1) = (-1)^{(k+1)+1}$ (k+1)(k+2)/2; true for all positive integers n by induction.

Question 7

(a) 5+9+2 = 16 = 4(4), divisible by 4 **B1**

(a) Answer: n = 1: 5 + 9 + 2 = 16, divisible by 4.

(b) assume true for n=k: $5^k+9^k+2 = 4m$ for some positive integer m **M1**

(b) consider $5^{k+1}+9^{k+1}+2 - 5(5^k+9^k+2)$ and expand **M1**

(b) simplify this difference to $4(9^k - 2)$ **dM1**

(b) hence $5^{k+1}+9^{k+1}+2 = 5(4m) + 4(9^k-2) = 4(5m+9^k-2)$, divisible by 4, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $5^{k+1}+9^{k+1}+2 = 4(5m+9^k-2)$; true for all positive integers n by induction.

Question 8

(a) $u_1 = 3 \times 3^0 - 1 = 2$ **B1**

(a) Answer: true for n = 1

(b) $u_{k+1} = 3u_k + 2 = 3(3 \times 3^{k-1} - 1) + 2$ **M1**

(b) expands: $3 \times 3^k - 3 + 2$ **M1**

(b) $2 - 3 = -1$, giving $3 \times 3^k - 1$ **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all n by induction (cso) **A1**

(b) Answer: proved by induction

Question 9

(a) f(1) = 5 x 1, divisible by 5 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 6^{k+1} - 1 = 6(6^k - 1) + 5 = 6f(k) + 5$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 10

(a) LHS = 6, RHS = 6, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 6

(b) assumes sum to $k = k(k + 1)(2k + 16)/6$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $(k + 1)((k + 1) + 1)(2(k + 1) + 16)/6$ by the simplification: $k(k + 1)(2k + 16)/6 + (k + 1)(k + 6) = (k + 1)[k(2k + 16) + 6(k + 6)]/6 = (k + 1)(k + 2)(2k + 18)/6$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 11

(a) LHS = 2, RHS = 2, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 2

(b) assumes sum to $k = 2k^2(k + 1)^2/4$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $2(k + 1)^2((k + 1) + 1)^2/4$ by the simplification: $2k^2(k + 1)^2/4 + 2(k + 1)^3 = 2(k + 1)^2[k^2 + 4(k + 1)]/4 = 2(k + 1)^2(k + 2)^2/4$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 12

(a) evaluates the formula at n = 1: ((4, 0), (3, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: $[(4^k - 1)] \times 4 + 3 = (4^{k+1} - 1)$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 13

(a) f(1) = 49 - 16 = 33, divisible by 33 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k+2} - 4^{2k+2} = 49f(k) + 4^{2k}(49 - 16) = 49f(k) + 33 \times 4^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 14

(a) f(1) = 7 x 1, divisible by 7 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 8^{k+1} - 1 = 8(8^k - 1) + 7 = 8f(k) + 7$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 15

(a) n = 1: 2(3) + 1(5) = 11 **B1**

(a) n = 2: 2(9) + 1(25) = 43 **B1**

(a) Answer: true for n = 1 and n = 2

(b) $u_{k+2} = 8u_{k+1} - 15u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $2 \times 3^k(8 \times 3 - 15) = 2 \times 3^{k+2}$ since $3^2 - 8 \times 3 + 15 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $2 \times 3^{k+2} + 1 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

(b) Answer: proved by induction

Question 16

(a) n = 1: 2(3) - 2(5) = -4 **B1**

(a) n = 2: 2(9) - 2(25) = -32 **B1**

(a) Answer: true for n = 1 and n = 2

(b) $u_{k+2} = 8u_{k+1} - 15u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $2 \times 3^k(8 \times 3 - 15) = 2 \times 3^{k+2}$ since $3^2 - 8 \times 3 + 15 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $2 \times 3^{k+2} - 2 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

(b) Answer: proved by induction

Question 17

(a) LHS = 1, RHS = 1, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 1

(b) assumes sum to k = $k^2(k + 1)^2/4$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $(k + 1)^2((k + 1) + 1)^2/4$ by the simplification: $k^2(k + 1)^2/4 + 1(k + 1)^3 = 1(k + 1)^2[k^2 + 4(k + 1)]/4 = 1(k + 1)^2(k + 2)^2/4$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 18

(a) f(1) = 49 - 25 = 24, divisible by 24 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k+2} - 5^{2k+2} = 49f(k) + 5^{2k}(49 - 25) = 49f(k) + 24 \times 5^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 19

(a) evaluates the formula at n = 1: ((2, 1), (0, 2)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: $2^k \times 1 + k2^{k-1} \times 2 = (k + 1)2^k$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 20

(a) f(1) = 49 - 36 = 13, divisible by 13 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 7^{2k+2} - 6^{2k+2} = 49f(k) + 6^{2k}(49 - 36) = 49f(k) + 13 \times 6^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 21

(a) n = 1: 1(2) + 1(5) = 7 **B1**

(a) n = 2: 1(4) + 1(25) = 29 **B1**

(a) Answer: true for n = 1 and n = 2

(b) $u_{k+2} = 7u_{k+1} - 10u_k$ and substitutes the assumptions **M1**

(b) collects the r1 terms: $1 \times 2^k(7 \times 2 - 10) = 1 \times 2^{k+2}$ since $2^2 - 7 \times 2 + 10 = 0$ **M1**

(b) similarly for the r2 terms, obtaining $1 \times 2^{k+2} + 1 \times 5^{k+2}$ **A1**

(b) true for n = 1, 2; true for n = k, k + 1 implies true for n = k + 2; so true for all positive integers n by induction (cso) **A1**

(b) Answer: proved by induction

Question 22

(a) evaluates the formula at n = 1: ((1, 2), (0, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k+1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: 1 x 2 + 2k x 1 = 2(k + 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 23

(a) $u_1 = 6 \times 2^0 - 3 = 3$ **B1**

(a) Answer: true for n = 1

(b) $u_{k+1} = 2u_k + 3 = 2(6 \times 2^{k-1} - 3) + 3$ **M1**

(b) expands: $6 \times 2^k - 6 + 3$ **M1**

(b) 3 - 6 = -3, giving $6 \times 2^k - 3$ **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all n by induction (cso) **A1**

(b) Answer: proved by induction

Question 24

(a) f(1) = 2 x 1, divisible by 2 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 3^{k + 1} - 1 = 3(3^k - 1) + 2 = 3f(k) + 2$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 25

(a) LHS = 2, RHS = 2, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 2

(b) assumes sum to k = $(k - 1)2^{k + 1} + 2$ and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains $((k + 1) - 1)2^{(k + 1) + 1} + 2$ by the simplification: $(k - 1)2^{k + 1} + 2 + (k + 1)2^{k + 1} = 2^{k + 1}(2k) + 2 = k\,2^{k + 2} + 2$ **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 26

(a) $u_1 = 5 \times 3^0 - 3 = 2$ **B1**

(a) Answer: true for n = 1

(b) $u_{k+1} = 3u_k + 6 = 3(5 \times 3^{k - 1} - 3) + 6$ **M1**

(b) expands: $5 \times 3^k - 9 + 6$ **M1**

(b) 6 - 9 = -3, giving $5 \times 3^k - 3$ **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all n by induction (cso) **A1**

(b) Answer: proved by induction

Question 27

(a) LHS = 3/2, RHS = 3/2, so true for n = 1 **B1**

(a) Answer: n = 1: both sides equal 3/2

(b) assumes sum to k = 3k/(k + 1) and adds the (k + 1)th term **M1**

(b) factorises or expands to simplify **M1**

(b) obtains 3(k + 1)/((k + 1) + 1) by the simplification: 3k/(k + 1) + 3/((k + 1)(k + 2)) = 3[k(k + 2) + 1]/((k + 1)(k + 2)) = 3(k + 1)/(k + 2) **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 28

(a) evaluates the formula at n = 1: ((1, 5), (0, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k + 1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (top right: 1 x 5 + 5k x 1 = 5(k + 1)) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 29

(a) evaluates the formula at n = 1: ((3, 0), (4, 1)) = M **B1**

(a) Answer: true for n = 1

(b) assumes M^k as the formula and writes $M^{k + 1} = M^k M$ **M1**

(b) carries out the matrix multiplication **M1**

(b) simplifies the entries (bottom left: $[2(3^k - 1)] \times 3 + 4 = 2(3^{k + 1} - 1)$) **A1**

(b) true for n = 1, true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 30

(a) f(1) = 25 - 16 = 9, divisible by 9 **B1**

(a) Answer: true for n = 1

(b) assumes f(k) = m x (an integer) for some positive integer k **M1**

(b) expresses f(k + 1) in terms of f(k): $f(k + 1) = 5^{2k + 2} - 4^{2k + 2} = 25f(k) + 4^{2k}(25 - 16) = 25f(k) + 9 \times 4^{2k}$ **M1**

(b) both terms are multiples of the divisor, so f(k + 1) is divisible **A1**

(b) true for n = 1, and true for n = k implies true for n = k + 1, so true for all positive integers n by induction (cso) **A1**

(b) Answer: result proved by induction

Question 31

(a) multiply M by M to get $M^2 = ((9, 0), (4, 1))$ **M1**

(a) $(3^2-1)/2 = 4$, matching the bottom-left entry **A1**

(a) Answer: $M^2 = ((9, 0), (4, 1))$

(b) assume true for n=k: $M^k = ((3^k,0),((3^k-1)/2,1))$, form $M^{k+1} = M^k * M$ **M1**

(b) carry out the matrix multiplication $M^k * M$ correctly **M1**

(b) simplify the bottom-left entry: $3(3^k-1)/2 + 1$ **dM1**

(b) simplify to $(3^{k+1}-1)/2$, and top-left to 3^{k+1}, cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $M^{k+1} = ((3^{k+1},0),((3^{k+1}-1)/2,1))$; true for all positive integers n by induction.

Question 32

(a) for n=1, LHS = cos(θ)+isin(θ) = RHS = cos(1.θ)+isin(1.θ), so true **B1**

(a) Answer: n = 1: both sides equal cos(θ) + i sin(θ).

(b) assume true for n=k: $(\cos\theta + i\sin\theta)^k = \cos(k\theta)+i\sin(k\theta)$ **M1**

(b) write $(\cos\theta+i\sin\theta)^{k+1} = (\cos(k\theta)+i\sin(k\theta))(\cos\theta+i\sin\theta)$ and expand **M1**

(b) collect real and imaginary parts using the addition formulae **dM1**

(b) simplify to $\cos((k+1)\theta) + i\sin((k+1)\theta)$ cso **A1**

(b) conclusion: true for n=1, true for n=k implies true for n=k+1, so true for all positive integers n by mathematical induction **A1**

(b) Answer: $(\cos\theta+i\sin\theta)^{k+1} = \cos((k+1)\theta)+i\sin((k+1)\theta)$; true for all positive integers n by induction (De Moivre's theorem).

(c) formula gives $\cos(0)+i\sin(0) = 1$, consistent with $z^0 = 1$ for any non-zero z **B1**

(c) Answer: $\cos(0) + i\sin(0) = 1$, consistent with $z^0 = 1$.