2.1 Further Statistics: Discrete Random Variables and Expectation: 1-hour Pack

AQA 7367 · Calculator allowed · about 60 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

Key facts: discrete random variables and expectation

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A discrete random variable X takes a countable set of values, each with an associated probability P(X = x), where the probabilities sum to 1. The expectation (mean) is E(X) = sum of x P(X = x), and the variance is Var(X) = E(X^2) - [E(X)]^2, where E(X^2) = sum of x^2 P(X = x). For constants a and b, expectation and variance obey the rules E(aX + b) = aE(X) + b and Var(aX + b) = a^2 Var(X); more generally, E(g(X)) = sum of g(x) P(X = x) for any function g. A discrete uniform distribution on the integers 1 to n, where every value is equally likely, has the standard results E(X) = (n + 1)/2 and Var(X) = (n^2 - 1)/12. If X1 and X2 are independent random variables, then E(X1 + X2) = E(X1) + E(X2) and Var(X1 + X2) = Var(X1) + Var(X2); by contrast, doubling a single variable (2X) multiplies its variance by 2^2 = 4, since scaling one variable and summing two independent copies are different operations. These results underpin every calculation in this worksheet.

1 A discrete random variable X takes values 2, 4, 6 only, with P(X = x) = kx for these values, where k is a constant.

(a) Show that k = 1/12. **(2)**

(b) Find E(X), giving your answer as an exact fraction. **(3)**

(c) Find Var(X), giving your answer as an exact fraction. **(4)**

(d) Hence find Var(6X - 2), giving your answer as an exact value. **(2)**

(Total for Question 1 is 11 marks)

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2 The independent discrete random variables X and Y each take the values 1, 2 and 3. P(X = 1), P(X = 2), P(X = 3) are 1/3, 2/9, 4/9 respectively, and P(Y = 1), P(Y = 2), P(Y = 3) are 4/9, 4/9, 1/9 respectively.

(a) Find P(X = Y). **(2)**

(b) Find P(X > Y). **(2)**

(c) Find E(X + Y) and Var(X - Y). **(4)**

(Total for Question 2 is 8 marks)

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3 The discrete random variable X has probability distribution P(X = x) = k(5 - x) for x = 1, 2, 3, 4 and 0 otherwise, where k is a constant.

(a) Show that k = 1/10. **(3)**

(b) Find E(X) and Var(X). **(4)**

(c) Find E(4X + 5) and Var(4X + 5). **(4)**

(Total for Question 3 is 11 marks)

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4 The independent discrete random variables X and Y each take the values 1, 2 and 3. P(X = 1), P(X = 2), P(X = 3) are 5/14, 3/7, 3/14 respectively, and P(Y = 1), P(Y = 2), P(Y = 3) are 4/11, 3/11, 4/11 respectively.

(a) Find P(X = Y). **(2)**

(b) Find P(X > Y). **(2)**

(c) Find E(X + Y) and Var(X - Y). **(4)**

(Total for Question 4 is 8 marks)

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5 A fairground 'Lucky Dip' bag at a fete in Cardiff contains tickets numbered 1 to 8, one of which is drawn at random. Let X be the number on the ticket drawn, so X has a discrete uniform distribution on {1, 2, ..., 8}. The prize money is modelled by $P = 3X^2 - 4X + 1$ (in pounds).

(a) State E(X) and Var(X), using the standard results for a discrete uniform distribution. **(2)**

(b) Find $E(X^2)$. **(3)**

(c) Hence find E(P). **(3)**

(d) The organiser wants E(P) ≤ 50 to keep the game profitable for the fete. State, with a reason, whether the current prize structure meets this requirement. **(2)**

(Total for Question 5 is 10 marks)

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Mark scheme · 2.1 Further Statistics: Discrete Random Variables and Expectation: 1-hour Pack

Question 1

(a) $k(2 + 4 + 6) = 1$, i.e. $12k = 1$ oe **M1**

(a) $k = 1/12$ cso **A1**

(a) Answer: $k = 1/12$

(b) $E(X) = 2(2/12) + 4(4/12) + 6(6/12)$ oe **M1**

(b) $= (4 + 16 + 36)/12 = 56/12$ **A1**

(b) $= 14/3$ cao (exact fraction, simplified) **A1**

(b) Answer: $E(X) = 14/3$

(c) $E(X^2) = 4(2/12) + 16(4/12) + 36(6/12)$ oe **M1**

(c) $E(X^2) = (8 + 64 + 216)/12 = 288/12 = 24$ **A1**

(c) $Var(X) = E(X^2) - [E(X)]^2 = 24 - (14/3)^2$ (dependent on their $E(X^2)$ and $E(X)$) **dM1**

(c) $= 20/9$ cao **A1**

(c) Answer: $Var(X) = 20/9$

(d) $Var(6X - 2) = 6^2\ Var(X) = 36(20/9)$ oe **M1**

(d) $= 80$ cao **A1**

(d) Answer: 80

Question 2

(a) sum of $P(X = i)P(Y = i)$ over $i = 1, 2, 3$, using independence **M1**

(a) $8/27$ **A1**

(a) Answer: 8/27

(b) adds $P(X = i)P(Y = j)$ over the cases $i > j$ **M1**

(b) $40/81$ **A1**

(b) Answer: 40/81

(c) $E(X + Y) = E(X) + E(Y)$ **M1**

(c) $34/9$ **A1**

(c) $Var(X - Y) = Var(X) + Var(Y)$ for independent variables **M1**

(c) $98/81$ **A1**

(c) Answer: $E(X + Y) = 34/9$, $Var(X - Y) = 98/81$

Question 3

(a) uses the sum of all probabilities $= 1$ **M1**

(a) forms the sum $P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4)$ in terms of k **M1**

(a) $k(10) = 1$, so $k = 1/10$ (cso) **A1**

(a) Answer: $k = 1/10$

(b) $E(X) =$ sum of $x\ P(X = x)$ **M1**

(b) $E(X) = 2$ **A1**

(b) $Var(X) = E(X^2) - [E(X)]^2$ **M1**

(b) $E(X^2) = 5$, $Var(X) = 1$ **A1**

(b) Answer: $E(X) = 2$, $Var(X) = 1$

(c) $E(aX + b) = aE(X) + b$ **M1**

(c) 13 **A1**

(c) $Var(aX + b) = a^2\ Var(X)$ **M1**

(c) 16 **A1**

(c) Answer: E = 13, Var = 16

Question 4

(a) sum of $P(X = i)P(Y = i)$ over $i = 1, 2, 3$, using independence **M1**

(a) $25/77$ **A1**

(a) Answer: 25/77

(b) adds $P(X = i)P(Y = j)$ over the cases $i > j$ **M1**

(b) $45/154$ **A1**

(b) Answer: 45/154

(c) $E(X + Y) = E(X) + E(Y)$ **M1**

(c) $27/7$ **A1**

(c) $Var(X - Y) = Var(X) + Var(Y)$ for independent variables **M1**

(c) $689/539$ **A1**

(c) Answer: $E(X + Y) = 27/7$, $Var(X - Y) = 689/539$

Question 5

(a) $E(X) = 4.5$ **B1**

(a) $Var(X) = 5.25$ $(= 63/12)$ **B1**

(a) Answer: $E(X) = 4.5$, $Var(X) = 5.25$

(b) rearranges $Var(X) = E(X^2) - [E(X)]^2$ to $E(X^2) = Var(X) + [E(X)]^2$ oe **M1**

(b) $E(X^2) = 5.25 + 4.5^2$ **A1**

(b) $= 25.5$ cao **A1**

(b) Answer: 25.5

(c) $E(P) = 3E(X^2) - 4E(X) + 1$ oe **M1**

(c) substitutes $3(25.5) - 4(4.5) + 1$ **A1**

(c) $= 59.5$ cao **A1**

(c) Answer: 59.5

(d) correctly states that the requirement is not met (No) **B1**

(d) gives a valid reason referencing the calculated value, e.g. since $E(P) = 59.5 > 50$ oe **B1**

(d) Answer: No; since E(P) = £59.5, which is greater than £50, the average prize paid out exceeds the organiser's target, so the game as currently structured is not sufficiently profitable.

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