2.1 Further Statistics: Discrete Random Variables and Expectation: Test Pack

AQA 7367 · Calculator allowed · about 65 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

Key facts: discrete random variables and expectation

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A discrete random variable X takes a countable set of values, each with an associated probability P(X = x), where the probabilities sum to 1. The expectation (mean) is E(X) = sum of x P(X = x), and the variance is Var(X) = E(X^2) - [E(X)]^2, where E(X^2) = sum of x^2 P(X = x). For constants a and b, expectation and variance obey the rules E(aX + b) = aE(X) + b and Var(aX + b) = a^2 Var(X); more generally, E(g(X)) = sum of g(x) P(X = x) for any function g. A discrete uniform distribution on the integers 1 to n, where every value is equally likely, has the standard results E(X) = (n + 1)/2 and Var(X) = (n^2 - 1)/12. If X1 and X2 are independent random variables, then E(X1 + X2) = E(X1) + E(X2) and Var(X1 + X2) = Var(X1) + Var(X2); by contrast, doubling a single variable (2X) multiplies its variance by 2^2 = 4, since scaling one variable and summing two independent copies are different operations. These results underpin every calculation in this worksheet.

1 A tutor, Mr Osei, uses a fair 10-sided die, numbered 1 to 10, to randomly assign the number of practice questions a student completes. Let X be the number shown.

(a) State the distribution of X and calculate E(X) and Var(X). **(3)**

(b) The tutor decides to instead set X + 1 practice questions, to guarantee at least one question. State E(X + 1) and Var(X + 1). **(2)**

(c) Explain, without further calculation, why Var(X + 1) = Var(X). **(1)**

(Total for Question 1 is 6 marks)

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2 A discrete random variable X takes values 1, 2, 3 and 4 only, with probability distribution given by $P(X = x) = k(5 - x)$ for $x = 1, 2, 3, 4$, where k is a constant.

(a) Show that $k = 0.1$. **(3)**

(b) Find E(X). **(2)**

(c) Find Var(X). **(3)**

(d) Find E(3X - 2) and Var(3X - 2). **(3)**

(Total for Question 2 is 11 marks)

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3 The number of defective components picked from a batch, X, has probability distribution:

x: 0, 1, 2

P(X = x): 0.6, 0.3, 0.1

The rework cost, in pounds, is modelled by $C = 20X^2 - 5X + 100$.

(a) Show that E(X) = 0.5. **(2)**

(b) Find $E(X^2)$. **(2)**

(c) Hence find E(C). **(3)**

(Total for Question 3 is 7 marks)

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4 The independent discrete random variables X and Y each take the values 1, 2 and 3. P(X = 1), P(X = 2), P(X = 3) are 3/7, 3/7, 1/7 respectively, and P(Y = 1), P(Y = 2), P(Y = 3) are 1/12, 5/12, 1/2 respectively.

(a) Find P(X = Y). **(2)**

(b) Find P(X > Y). **(2)**

(c) Find E(X + Y) and Var(X - Y). **(4)**

(Total for Question 4 is 8 marks)

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5 The discrete random variable X has probability distribution $P(X = x) = k/x$ for $x = 1, 2, 3, 4$ and 0 otherwise, where k is a constant.

(a) Show that $k = 12/25$. **(3)**

(b) Find E(X) and Var(X). **(4)**

(c) Find E(3X - 3) and Var(3X - 3). **(4)**

(Total for Question 5 is 11 marks)

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6 In a game, a single play gives winnings X pounds with E(X) = 3 and Var(X) = 2. A player plays the game twice, independently; let T = X1 + X2 be the total winnings from the two plays. Separately, consider W = 2X, the winnings if a rule change simply doubled whatever a single play produced.

(a) Find E(T) and Var(T). **(3)**

(b) Find E(W) and Var(W). **(3)**

(c) Explain why Var(T) is not equal to Var(W), even though E(T) = E(W). **(2)**

(Total for Question 6 is 8 marks)

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Mark scheme · 2.1 Further Statistics: Discrete Random Variables and Expectation: Test Pack

Question 1

(a) X has a discrete uniform distribution on {1, 2, ..., 10} **B1**

(a) E(X) = 5.5 **B1**

(a) Var(X) = 8.25 (= 99/12) **B1**

(a) Answer: E(X) = 5.5, Var(X) = 8.25

(b) E(X + 1) = 6.5 **B1**

(b) Var(X + 1) = 8.25 (unchanged) **B1**

(b) Answer: E(X + 1) = 6.5, Var(X + 1) = 8.25

(c) adding a constant shifts every possible value of X by the same fixed amount, so the spread of the values about the mean is unchanged oe **B1**

(c) Answer: Adding a constant translates the whole distribution but does not stretch or compress it, so the variance (spread) is unaffected.

Question 2

(a) sum of probabilities = 1, leading to k(4 + 3 + 2 + 1) = 1 oe **M1**

(a) 10k = 1 oe **A1**

(a) k = 0.1 cso **A1**

(a) Answer: k = 0.1

(b) sum x P(X = x) = 1(0.4) + 2(0.3) + 3(0.2) + 4(0.1) oe **M1**

(b) E(X) = 2 cao **A1**

(b) Answer: E(X) = 2

(c) $E(X^2) = 1^2(0.4) + 2^2(0.3) + 3^2(0.2) + 4^2(0.1)$ oe **M1**

(c) $E(X^2) = 5$ **A1**

(c) $Var(X) = E(X^2) - [E(X)]^2 = 5 - 2^2 = 1$ cao **A1**

(c) Answer: Var(X) = 1

(d) E(3X - 2) = 3E(X) - 2 oe **M1**

(d) E(3X - 2) = 4 **A1**

(d) $Var(3X - 2) = 3^2$ Var(X) = 9 cao (ft their Var(X)) **B1**

(d) Answer: E(3X - 2) = 4, Var(3X - 2) = 9

Question 3

(a) 0(0.6) + 1(0.3) + 2(0.1) oe **M1**

(a) = 0.5 cso **A1**

(a) Answer: E(X) = 0.5

(b) 0(0.6) + 1(0.3) + 4(0.1) oe **M1**

(b) = 0.7 cao **A1**

(b) Answer: 0.7

(c) $E(C) = 20E(X^2) - 5E(X) + 100$ oe **M1**

(c) substitutes 20(0.7) - 5(0.5) + 100 **A1**

(c) = 111.5 cao **A1**

(c) Answer: 111.5

Question 4

(a) sum of P(X = i)P(Y = i) over i = 1, 2, 3, using independence **M1**

(a) 2/7 **A1**

(a) Answer: 2/7

(b) adds P(X = i)P(Y = j) over the cases i > j **M1**

(b) 3/28 **A1**

(b) Answer: 3/28

(c) E(X + Y) = E(X) + E(Y) **M1**

(c) 347/84 **A1**

(c) Var(X - Y) = Var(X) + Var(Y) for independent variables **M1**

(c) 6347/7056 **A1**

(c) Answer: E(X + Y) = 347/84, Var(X - Y) = 6347/7056

Question 5

(a) uses the sum of all probabilities = 1 **M1**

(a) forms the sum P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) in terms of k **M1**

(a) k(25/12) = 1, so k = 12/25 (cso) **A1**

(a) Answer: k = 12/25

(b) E(X) = sum of x P(X = x) **M1**

(b) E(X) = 48/25 **A1**

(b) $Var(X) = E(X^2) - [E(X)]^2$ **M1**

(b) $E(X^2) = 24/5$, Var(X) = 696/625 **A1**

(b) Answer: E(X) = 48/25, Var(X) = 696/625

(c) E(aX + b) = aE(X) + b **M1**

(c) 69/25 **A1**

(c) $Var(aX + b) = a^2$ Var(X) **M1**

(c) 6264/625 **A1**

(c) Answer: E = 69/25, Var = 6264/625

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Question 6

(a) E(T) = E(X1) + E(X2) = 2(3) oe **M1**

(a) E(T) = 6 **A1**

(a) since the plays are independent, Var(T) = Var(X1) + Var(X2) = 2(2) = 4 **B1**

(a) Answer: E(T) = 6, Var(T) = 4

(b) uses E(W) = 2E(X) and Var(W) = 2^2 Var(X) **M1**

(b) E(W) = 6 **A1**

(b) Var(W) = 8 cao **A1**

(b) Answer: E(W) = 6, Var(W) = 8

(c) recognises that T is the sum of two independent random variables, so its variance is the sum of the individual variances (2 + 2 = 4) oe **B1**

(c) recognises that W scales a single variable by a constant, so its variance scales by the square of that constant (2^2 x 2 = 8), which is a different operation from summing independent copies oe **B1**

(c) Answer: Var(T) = 4 because summing two independent variables adds their variances; Var(W) = 8 because scaling a single variable by 2 multiplies its variance by 2^2 = 4. The two processes model different situations (two independent plays versus one play's result doubled), so despite equal means, the spreads differ.

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