2.5 Further Statistics: Discrete Distributions Depth: Test Pack

Total Marks

Calculator allowed · about 60 minutes

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

Key facts: discrete distributions and probability generating functions

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For X ~ Po(lambda), P(X = x) = e^(-lambda) * lambda^x / x!, for x = 0, 1, 2, ..., with E(X) = lambda and Var(X) = lambda. If X ~ Po(lambda1) and Y ~ Po(lambda2) are independent, then X + Y ~ Po(lambda1 + lambda2).

For X ~ Geo(p), the number of independent trials up to and including the first success (write q = 1 - p), P(X = x) = q^(x-1) * p, for x = 1, 2, 3, ..., with E(X) = 1/p and Var(X) = q/p^2. The geometric distribution has the memoryless property: P(X > m+n | X > m) = P(X > n) for positive integers m and n.

For X ~ NB(r, p), the number of independent trials up to and including the rth success, P(X = x) = C(x-1, r-1) * p^r * q^(x-r), for x = r, r+1, r+2, ..., with E(X) = r/p and Var(X) = rq/p^2. X can be written as the sum of r independent Geo(p) random variables.

The probability generating function of a discrete random variable X taking non-negative integer values is G_X(t) = E(t^X) = sum of P(X = x) * t^x. It satisfies G_X(1) = 1, E(X) = G_X'(1), and Var(X) = G_X''(1) + G_X'(1) - [G_X'(1)]^2. If X and Y are independent random variables, then G_(X+Y)(t) = G_X(t) * G_Y(t).

1 A call centre in Norwich receives incoming telephone calls singly and at random, independently of each other, at a constant average rate of 3.6 calls per 10-minute period. Let X be the number of calls received during a randomly chosen 10-minute period, so that X ~ Po(3.6).

(a) Find P(X = 4), giving your answer to 3 significant figures. **(2)**

(b) Find P(X ≥ 2), giving your answer to 3 significant figures. **(2)**

(Total for Question 1 is 4 marks)

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2 The random variable X has distribution B(80, 0.05).

(a) State two conditions under which a Poisson distribution is a suitable approximation to a binomial distribution, and give the parameter of the approximating distribution. **(2)**

(b) Use the binomial distribution to find P(X = 3), and use the Poisson approximation to find the same probability, each to 4 significant figures. **(4)**

(c) Find the percentage error in using the Poisson approximation here, to 2 significant figures. **(2)**

(Total for Question 2 is 8 marks)

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3 The discrete random variable X has probability distribution P(X = x) = k(5 - x) for x = 1, 2, 3, 4 and 0 otherwise, where k is a constant.

(a) Show that k = 1/10. **(3)**

(b) Find E(X) and Var(X). **(4)**

(c) Find E(5X + 5) and Var(5X + 5). **(4)**

(Total for Question 3 is 11 marks)

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4 A player attempts a task repeatedly until the first success. Each attempt succeeds with probability 0.4, independently of the others. Let X be the number of the attempt on which the first success occurs, so X ~ Geo(0.4).

(a) Find P(X = 2) and P(X > 2), each to 3 significant figures. **(4)**

(b) Find E(X) and Var(X). **(4)**

(c) Find the least number of attempts n such that the probability of at least one success within n attempts exceeds 0.95. **(4)**

(Total for Question 4 is 12 marks)

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5 Small pieces of grit land on a stretch of pavement in Aberdeen singly and at random, independently of each other, at a constant average rate of 0.6 pieces of grit per metre length.

(a) Let X be the number of pieces of grit found on a randomly chosen 5 metre length of pavement. State the distribution of X, including the value of any parameter(s). **(2)**

(b) Find P(X = 3), giving your answer to 3 significant figures. **(2)**

(c) Find P(X ≥ 1), giving your answer to 3 significant figures. **(2)**

(d) Two non-overlapping stretches of pavement, of lengths 5 metres and 8 metres, are considered. Using a suitable property of the Poisson distribution, find the probability that the total number of pieces of grit found on the combined 13 metre stretch is at most 5, giving your answer to 3 significant figures. **(5)**

(Total for Question 5 is 11 marks)

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Mark scheme · 2.5 Further Statistics: Discrete Distributions Depth: Test Pack

Question 1

(a) correct use of $P(X=4) = e^{-3.6} * 3.6^4 / 4!$ (or calculator Poisson pdf), oe	**M1**
(a) awrt 0.191	**A1**
(a) Answer: P(X = 4) = awrt 0.191	
(b) uses $P(X\geq2) = 1 - P(X=0) - P(X=1)$, oe	**M1**
(b) awrt 0.874	**A1**
(b) Answer: P(X ≥ 2) = awrt 0.874	

Question 2

(a) n is large and p is small	**B1**
(a) X is approximately Po(80 x 0.05) = Po(4)	**B1**
(a) Answer: Po(4)	
(b) $80C3 \times 0.05^3 \times 0.95^{77}$	**M1**
(b) 0.1978	**A1**
(b) $e^{-4}\ 4^3/3!$	**M1**
(b) 0.1954	**A1**
(b) Answer: binomial 0.1978, Poisson 0.1954	
(c) |approximate - exact|/exact x 100	**M1**
(c) 1.2%	**A1**
(c) Answer: 1.2%	

Question 3

(a) uses the sum of all probabilities = 1	**M1**
(a) forms the sum P(X = 1) + P(X = 2) + P(X = 3) + P(X = 4) in terms of k	**M1**
(a) k(10) = 1, so k = 1/10 (cso)	**A1**
(a) Answer: k = 1/10	
(b) E(X) = sum of x P(X = x)	**M1**
(b) E(X) = 2	**A1**
(b) $Var(X) = E(X^2) - [E(X)]^2$	**M1**
(b) $E(X^2) = 5$, Var(X) = 1	**A1**
(b) Answer: E(X) = 2, Var(X) = 1	
(c) E(aX + b) = aE(X) + b	**M1**
(c) 15	**A1**
(c) $Var(aX + b) = a^2\ Var(X)$	**M1**
(c) 25	**A1**
(c) Answer: E = 15, Var = 25	

Question 4

(a) P(X = 2) = (1 - 0.4) x 0.4	**M1**
(a) 0.240	**A1**
(a) $P(X > 2) = (1 - 0.4)^2$	**M1**
(a) 0.360	**A1**
(a) Answer: 0.240; 0.360	
(b) E(X) = 1/p	**M1**
(b) 5/2	**A1**
(b) $Var(X) = (1 - p)/p^2$	**M1**
(b) 3.750	**A1**
(b) Answer: E(X) = 2.500, Var(X) = 3.750	
(c) $P(X \leq n) = 1 - (1 - 0.4)^n > 0.95$	**M1**
(c) $(1 - 0.4)^n < 0.05$, and takes logs (changing the inequality)	**M1**
(c) n > 5.864	**A1**
(c) n = 6	**A1**
(c) Answer: n = 6	

Question 5

(a) uses parameter = rate * length = 0.6 * 5	**M1**
(a) X ~ Po(3), cao	**A1**
(a) Answer: X ~ Po(3)	
(b) correct use of $P(X=3) = e^{-3} * 3^3/3!$ with λ = 3	**M1**
(b) awrt 0.224	**A1**
(b) Answer: P(X = 3) = awrt 0.224	
(c) uses $P(X\geq1) = 1 - P(X=0)$, oe	**M1**
(c) awrt 0.950	**A1**
(c) Answer: P(X ≥ 1) = awrt 0.950	
(d) recognises the counts on the two non-overlapping stretches are independent Poisson variables, with parameters 0.6*5=3 and 0.6*8=4.8	**M1**
(d) uses the additive property (established in Q7): total W ~ Po(3+4.8) = Po(7.8), oe	**M1**
(d) sums the Poisson probabilities P(W=0) + P(W=1) + ... + P(W=5), or uses a calculator cumulative Poisson function	**M1**
(d) correct unsimplified sum, or calculator value, of awrt 0.2103	**A1**
(d) awrt 0.210	**A1**
(d) Answer: P(W ≤ 5) = awrt 0.210	

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