1.1 Pure: Proof

EDEXCEL 9MA0 · Calculator allowed · about 230 minutes · Write-in edition

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 State whether 78 is odd or even, and write it in the form 2n for an integer n.

(Total for Question 1 is 1 mark)

2 Fully factorise $n^2 - n$.

(Total for Question 2 is 1 mark)

3 Expand and fully simplify $(2k + 1)(2k - 1)$.

(Total for Question 3 is 1 mark)

4 State the general algebraic form used to represent an even integer, and hence write 132 in that form.

(Total for Question 4 is 1 mark)

5 This question explores proof by exhaustion and its limitations, using the statement: for every positive integer n, n^3+n is even.

(a) Prove by exhaustion that n^3+n is even for every integer n such that $1 \leq n \leq 4$. **(4)**

(b) Prove, by considering the cases n even and n odd, that n^3+n is even for every positive integer n. **(4)**

(c) Explain why the working in part (a) alone does not prove that n^3+n is even for every positive integer n. **(2)**

(Total for Question 5 is 10 marks)

6 In each part, find a counter-example to disprove the given statement.

(a) Disprove the statement: "For every real number x, $x^2 \geq x$." **(2)**

(b) Disprove the statement: "For every integer n, if n is prime then n is odd." **(2)**

(c) Disprove the statement: "For all real numbers a and b, $(a+b)^2 = a^2+b^2$." **(2)**

(d) Disprove the statement: "For $0 \leq \theta \leq 360$ (degrees), $\sin(\theta) \leq \cos(\theta)$ implies $\theta \leq 45$." **(2)**

(Total for Question 6 is 8 marks)

7 Prove that the sum of any two even integers is always even.

(Total for Question 7 is 2 marks)

8 Prove that the product of any two odd integers is always odd.

(Total for Question 8 is 2 marks)

9 Give one value of n that disproves the statement: "$n^2 + 1$ is even for every positive integer n."

(Total for Question 9 is 1 mark)

10 This question involves proof by deduction using cases.

(a) Prove that the product of any two consecutive integers, n(n+1), is always even. **(3)**

(b) Hence prove that n^2+n+1 is always odd, for any integer n. **(3)**

(Total for Question 10 is 6 marks)

11 Prove that $n^2 - n + 3$ is odd for every integer n.

(Total for Question 11 is 3 marks)

12 Priya claims: "For every positive integer n, the value of $n^2 + n + 2$ is even."

(a) Verify Priya's claim for n = 1, n = 2 and n = 3. **(2)**

(b) Prove that Priya's claim is true for every positive integer n. **(2)**

(Total for Question 12 is 4 marks)

13 $f(x) = x^2 + 2x + 9$.

(a) Write $f(x)$ in the form $a(x + p)^2 + q$, where a, p and q are constants to be found. **(3)**

(b) Hence prove that $f(x) > 0$ for all real values of x. **(2)**

(c) State the coordinates of the minimum point of the curve $y = f(x)$, and explain why the curve does not meet the x-axis. **(2)**

(Total for Question 13 is 7 marks)

14 This question is about proof by contradiction.

(a) Prove by contradiction that there are no integers x and y such that $4x + 10y = 5$. **(4)**

(b) Find one pair of integers x and y such that $4x + 10y = 6$. **(2)**

(Total for Question 14 is 6 marks)

15 This question is about disproof by counterexample and proof by deduction.

(a) A student claims: 'If $x > y$ then $x^2 > y^2$ for all real numbers x and y.' Find a counterexample to this claim. **(3)**

(b) State, with a reason, a condition on x and y under which the claim is true. **(2)**

(Total for Question 15 is 5 marks)

16 $f(x) = 3x^2 + 30x + 82$.

(a) Write $f(x)$ in the form $a(x + p)^2 + q$, where a, p and q are constants to be found. **(3)**

(b) Hence prove that $f(x) > 0$ for all real values of x. **(2)**

(c) State the coordinates of the minimum point of the curve $y = f(x)$, and explain why the curve does not meet the x-axis. **(2)**

(Total for Question 16 is 7 marks)

17 This question is about proof by contradiction.

(a) Prove by contradiction that there are no integers x and y such that $2x + 8y = 11$. **(4)**

(b) Find one pair of integers x and y such that $2x + 8y = 6$. **(2)**

(Total for Question 17 is 6 marks)

18 This question is about proof by algebraic argument.

(a) Prove that, for any integer n, $n(n + 1)(n + 2)(n + 3) + 1$ is the square of an integer. **(4)**

(b) Use this result to write 1681 as the square of an integer. **(2)**

(Total for Question 18 is 6 marks)

19 This question is about disproof by counterexample and proof by deduction.

(a) A student claims that $5n + 11$ is prime for every positive integer n. Show that this is not true. **(3)**

(b) Prove that $5n + 11$ is divisible by 11 whenever n is a multiple of 11. **(3)**

(Total for Question 19 is 6 marks)

20 This question is about disproof by counterexample and proof by deduction.

(a) A student claims that $4n + 13$ is prime for every positive integer n. Show that this is not true. **(3)**

(b) Prove that $4n + 13$ is divisible by 13 whenever n is a multiple of 13. **(3)**

(Total for Question 20 is 6 marks)

21 $f(x) = x^2 - 10x + 30$.

(a) Write $f(x)$ in the form $a(x + p)^2 + q$, where a, p and q are constants to be found. **(3)**

(b) Hence prove that $f(x) > 0$ for all real values of x. **(2)**

(c) State the coordinates of the minimum point of the curve $y = f(x)$, and explain why the curve does not meet the x-axis. **(1)**

(Total for Question 21 is 6 marks)

22 This question is about proof by contradiction.

(a) Prove by contradiction that $\sqrt{5}$ is irrational. **(4)**

(b) Hence prove that $5 + \sqrt{5}$ is irrational. **(2)**

(Total for Question 22 is 6 marks)

23 This question is about proof by algebraic argument.

(a) Prove that the sum of the squares of any two consecutive integers is odd. **(4)**

(b) Verify the result of part (a) for the integers 5 and 6. **(2)**

(Total for Question 23 is 6 marks)

24 This question is about proof by algebraic argument.

(a) Prove that, for any integer n, $n(n + 1)(n + 2)(n + 3) + 1$ is the square of an integer. **(4)**

(b) Use this result to write 121 as the square of an integer. **(2)**

(Total for Question 24 is 6 marks)

25 $f(x) = 2x^2 - 4x + 3$.

(a) Write $f(x)$ in the form $a(x + p)^2 + q$, where a, p and q are constants to be found. **(3)**

(b) Hence prove that $f(x) > 0$ for all real values of x. **(2)**

(c) State the coordinates of the minimum point of the curve $y = f(x)$, and explain why the curve does not meet the x-axis. **(1)**

(Total for Question 25 is 6 marks)

26 This question is about proof by contradiction.

(a) Prove by contradiction that $\sqrt{3}$ is irrational. **(4)**

(b) Hence prove that $3 + \sqrt{3}$ is irrational. **(2)**

(Total for Question 26 is 6 marks)

27 This question is about proof by algebraic argument.

(a) Prove that, for any integer n, $n(n + 1)(n + 2)(n + 3) + 1$ is the square of an integer. **(4)**

(b) Use this result to write 841 as the square of an integer. **(2)**

(Total for Question 27 is 6 marks)

28 This question involves an identity established using the compound angle formulae.

(a) Show that $\sin(\theta+60) + \sin(\theta-60) = \sin(\theta)$, where θ is measured in degrees. **(4)**

(b) Hence solve, for $0 \leq \theta < 360$ (degrees), the equation $\sin(\theta+60) + \sin(\theta-60) = 0.5$. **(3)**

(Total for Question 28 is 7 marks)

29 This question involves proof using algebraic inequalities.

(a) Prove that $x^2+y^2 \geq 2xy$ for all real numbers x and y. **(3)**

(b) Hence, by substituting x=a and y=b into the result of part (a), prove that $a/b + b/a \geq 2$ for all real a, b > 0. **(3)**

(Total for Question 29 is 6 marks)

30 $f(x) = x^3 + 6x^2 + 15x - 2$ for all real x.

(a) Find f'(x). **(2)**

(b) Show that f'(x) can be written in the form $3[(x+2)^2+1]$, and hence show that f'(x) > 0 for all real values of x. **(3)**

(c) State the conclusion that follows about the function f. **(1)**

(Total for Question 30 is 6 marks)

31 Prove that $x^2 + 4y^2 \geq 4xy$ for all real numbers x and y.

(Total for Question 31 is 3 marks)

32 This question builds a proof that $\sqrt{2}$ is irrational.

(a) Prove that if n^2 is even, where n is an integer, then n is even. **(2)**

(b) Prove, by contradiction, that $\sqrt{2}$ is irrational. **(7)**

(Total for Question 32 is 9 marks)

33 This question proves that n^3-n is divisible by 6 for every positive integer n.

(a) Fully factorise n^3-n. **(2)**

(b) Explain why (n-1)n(n+1) is always divisible by 2. **(2)**

(c) Explain why (n-1)n(n+1) is always divisible by 3. **(2)**

(d) Hence prove that n^3-n is divisible by 6 for every positive integer n. **(2)**

(Total for Question 33 is 8 marks)

34 This question proves, by contradiction, that there are infinitely many prime numbers.

(a) State the assumption made at the start of a proof by contradiction that there are infinitely many primes. **(1)**

(b) Let $N = p_1 \times p_2 \times ... \times p_k + 1$. Explain why none of $p_1, p_2, ..., p_k$ divides N. **(2)**

(c) Explain why this leads to a contradiction. **(2)**

(d) State the conclusion that follows from this contradiction. **(1)**

(e) Illustrate the argument using the first three primes, 2, 3 and 5. Calculate N = 2x3x5+1, verify that N is not divisible by 2, 3 or 5, and identify the new prime this reveals. **(3)**

(f) Calculate N for the first six primes, 2, 3, 5, 7, 11 and 13. Given that N = 59 x 509, explain why this shows the argument in parts (a) to (d) still works even when N itself is not prime. **(2)**

(Total for Question 34 is 11 marks)

Mark scheme · 1.1 Pure: Proof

Question 1

even, 78 = 2(39), n = 39 **B1**

Answer: Even; n = 39

Question 2

n(n - 1) cao **B1**

Answer: n(n - 1)

Question 3

$4k^2$ - 1 cao **B1**

Answer: $4k^2$ - 1

Question 4

2n, with n = 66 **B1**

Answer: 2n, n = 66

Question 5

(a) n=1: $1^3+1=2$, even **B1**

(a) n=2: $2^3+2=10$, even **B1**

(a) n=3: $3^3+3=30$, even **B1**

(a) n=4: $4^3+4=68$, even, and all four cases $1 \le n \le 4$ have been checked, cao **B1**

(a) Answer: n=1,2,3,4 give 2, 10, 30, 68, all even

(b) case n=2k: substitutes to get $n^3+n=8k^3+2k$ **M1**

(b) writes $8k^3+2k=2(4k^3+k)$, a multiple of 2, hence even **A1**

(b) case n=2k+1: substitutes and factorises $n^3+n=(2k+1)[(2k+1)^2+1]=(2k+1)(4k^2+4k+2)$ **M1**

(b) writes this as $2(2k+1)(2k^2+2k+1)$, a multiple of 2, hence even; since every integer is even or odd, this proves the result for all positive integers, cso **A1**

(b) Answer: Proof: n^3+n is even for all positive integers n

(c) states that part (a) only checks a finite number of cases (n=1 to 4), but the claim is about all positive integers, an infinite set **B1**

(c) states that a proof by exhaustion must check every case in the set, which is impossible for an infinite set, so a general argument (as in part (b)) is needed instead **B1**

(c) Answer: Checking 4 cases cannot establish a result for infinitely many integers

Question 6

(a) chooses a valid value of x with $0<x<1$, e.g. x=0.5, oe **B1**

(a) shows $x^2<x$ for this value, e.g. $0.5^2=0.25<0.5$, so the statement is false, cao **A1**

(a) Answer: x=0.5 gives 0.25 < 0.5, so the statement is false

(b) states n=2 **B1**

(b) justifies that 2 is prime and even, contradicting the statement, cao **A1**

(b) Answer: n=2 is prime and even, so the statement is false

(c) chooses valid values with a and b both non-zero, e.g. a=1, b=1, oe **B1**

(c) shows $(1+1)^2=4$ but $1^2+1^2=2$, and 4 is not equal to 2, so the statement is false, cao **A1**

(c) Answer: a=1, b=1 gives 4 not equal to 2, so the statement is false

(d) chooses a valid value of θ with θ>45 in the given range, e.g. θ=350, oe **B1**

(d) shows sin(350) is awrt -0.17 and cos(350) is awrt 0.98, so $\sin(\theta) \le \cos(\theta)$ holds, yet θ=350 is not ≤45, so the statement is false, cao **A1**

(d) Answer: θ=350 gives sin(θ) awrt -0.17 ≤ cos(θ) awrt 0.98, but θ>45, so the statement is false

Question 7

writes the two even integers as 2a and 2b, for integers a and b, and forms the sum 2a + 2b **M1**

writes the sum as 2(a + b), a multiple of 2, hence even, cso **A1**

Answer: Proof: 2a + 2b = 2(a + b), which is even

Question 8

writes the two odd integers as 2p + 1 and 2q + 1, and expands (2p + 1)(2q + 1) = 4pq + 2p + 2q + 1 **M1**

writes this as 2(2pq + p + q) + 1, a multiple of 2 plus 1, hence odd, cso **A1**

Answer: Proof: (2p+1)(2q+1) = 2(2pq+p+q)+1, which is odd

Question 9

n = 2 (gives 5, which is odd), or any other even n, oe **B1**

Answer: n = 2, since $2^2 + 1 = 5$ is odd

Question 10

(a) considers the case n=2k (n even) and writes n(n+1)=2k(2k+1), a multiple of 2 **M1**

(a) considers the case n=2k+1 (n odd) and writes n(n+1)=(2k+1)(2k+2)=2(2k+1)(k+1), a multiple of 2 **A1**

(a) since every integer n is even or odd, and the product is a multiple of 2 in both cases, n(n+1) is always even, cso **A1**

(a) Answer: Proof: n(n+1) is always even

(b) recognises $n^2+n=n(n+1)$, oe **M1**

(b) uses the result from part (a) that n(n+1) is even, ft **A1**

(b) states that even + 1 is odd, so n^2+n+1 is always odd, cso **A1**

(b) Answer: Proof: n^2+n+1 = n(n+1)+1 = even+1 = odd

Question 11

factorises $n^2 - n = n(n - 1)$ **M1**

states that n and n-1 are consecutive integers, so exactly one of them is even, hence n(n-1) is always even **A1**

cso: since n(n-1) is even, n(n-1)+3 = even + odd = odd for every integer n **A1**

Answer: Proof: $n^2 - n + 3 = n(n-1) + 3$ = even + 3, which is odd

Question 12

(a) correctly evaluates n=1: 4, n=2: 8, n=3: 14 **B1**

(a) states that all three values are even **B1**

(a) Answer: n=1: 4, n=2: 8, n=3: 14, all even

(b) recognises $n^2 + n = n(n+1)$, the product of two consecutive integers, so n(n+1) is always even **M1**

(b) cso: $n^2+n+2 = n(n+1) + 2$ = even + even = even for every positive integer n **A1**

(b) Answer: Proof: $n^2+n+2 = n(n+1)+2$, and n(n+1) is always even, so the sum is always even

Question 13

(a) begins to complete the square: $(x + p)^2$ with p = b/2 **M1**

(a) a = 1, p = 1 **A1**

(a) q = 8 **A1**

(a) Answer: $f(x) = (x + 1)^2 + 8$

(b) states that a squared term is non-negative **M1**

(b) so $f(x) \geq 8 > 0$ (cso) **A1**

(b) Answer: minimum value 8 > 0

(c) (-1, 8) **B1**

(c) the minimum value 8 is above the x-axis, so there are no real roots **B1**

(c) Answer: (-1, 8)

Question 14

(a) assumes there are integers x and y with 4x + 10y = 5 **M1**

(a) notes the left side is even, since 4x + 10y = 2(2x + 5y) **M1**

(a) but 5 is odd, a contradiction **A1**

(a) so no such integers exist (cso) **A1**

(a) Answer: no integer solutions

(b) systematic trial or use of the Euclidean algorithm **M1**

(b) a valid pair, for example x = -6, y = 3 **A1**

(b) Answer: x = -6, y = 3

Question 15

(a) chooses a negative value for y, or negative values for both **M1**

(a) for example x = -3, y = -4: x > y since -3 > -4 **M1**

(a) $x^2 = 9$ and $y^2 = 16$, and 9 < 16, so the claim is false **A1**

(a) Answer: x = -3, y = -4: x > y but $x^2 = 9 < y^2 = 16$

(b) x > y > 0 (or both non-negative) **B1**

(b) since squaring is an increasing function for positive numbers $(x^2 - y^2 = (x - y)(x + y) > 0)$ **B1**

(b) Answer: true when x and y are both positive (x + y > 0)

Question 16

(a) takes out the factor 3 **M1**

(a) a = 3, p = 5 **A1**

(a) q = 7 **A1**

(a) Answer: $f(x) = 3(x + 5)^2 + 7$

(b) states that a squared term is non-negative **M1**

(b) so $f(x) \geq 7 > 0$ (cso) **A1**

(b) Answer: minimum value 7 > 0

(c) (-5, 7) **B1**

(c) the minimum value 7 is above the x-axis, so there are no real roots **B1**

(c) Answer: (-5, 7)

Question 17

(a) assumes there are integers x and y with 2x + 8y = 11 **M1**

(a) notes the left side is even, since 2x + 8y = 2(x + 4y) **M1**

(a) but 11 is odd, a contradiction **A1**

(a) so no such integers exist (cso) **A1**

(a) Answer: no integer solutions

(b) systematic trial or use of the Euclidean algorithm **M1**

(b) a valid pair, for example x = 3, y = 0 **A1**

(b) Answer: x = 3, y = 0

Question 18

(a) pairs the factors: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$ **M1**

(a) lets $w = n^2 + 3n$ and writes the expression as $w(w + 2) + 1$ **M1**

(a) $w^2 + 2w + 1 = (w + 1)^2$ **A1**

(a) states $(n^2 + 3n + 1)^2$ with $n^2 + 3n + 1$ an integer (cso) **A1**

(a) Answer: $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$

(b) identifies n = 5 **M1**

(b) 41^2 **A1**

(b) Answer: 41^2

Question 19

(a) tries n = 11 (or other values until a composite appears) **M1**

(a) 5(11) + 11 = 66 **A1**

(a) 66 = 11 x 6, which is not prime, so the claim is false **A1**

(a) Answer: n = 11 gives 66 = 11 x 6

(b) writes n = 11m **M1**

(b) 5(11m) + 11 = 11(5m + 1) **A1**

(b) which is a multiple of 11 (cso) **A1**

(b) Answer: 11(5m + 1)

Question 20

(a) tries n = 13 (or other values until a composite appears) **M1**

(a) 4(13) + 13 = 65 **A1**

(a) 65 = 13 x 5, which is not prime, so the claim is false **A1**

(a) Answer: n = 13 gives 65 = 13 x 5

(b) writes n = 13m **M1**

(b) 4(13m) + 13 = 13(4m + 1) **A1**

(b) which is a multiple of 13 (cso) **A1**

(b) Answer: 13(4m + 1)

Question 21

(a) begins to complete the square: $(x + p)^2$ with $p = b/2$ **M1**

(a) $a = 1$, $p = -5$ **A1**

(a) $q = 5$ **A1**

(a) Answer: $f(x) = (x - 5)^2 + 5$

(b) states that a squared term is non-negative **M1**

(b) so $f(x) \geq 5 > 0$ (cso) **A1**

(b) Answer: minimum value $5 > 0$

(c) the minimum value 5 is above the x-axis, so there are no real roots **B1**

(c) Answer: (5, 5)

Question 22

(a) assumes $\sqrt{5} = a/b$ where a and b are integers with no common factor **M1**

(a) squares to obtain $a^2 = 5b^2$, so a^2 is a multiple of 5 **M1**

(a) deduces that a is a multiple of 5, $a = 5c$, and substitutes to obtain $5b^2 = 25c^2$, so $b^2 = 5c^2$ **A1**

(a) hence b is also a multiple of 5, contradicting that a and b have no common factor, so $\sqrt{5}$ is irrational (cso) **A1**

(a) Answer: $\sqrt{5}$ is irrational

(b) assumes $5 + \sqrt{5} = r$ with r rational **M1**

(b) then $\sqrt{5} = r - 5$ is rational, contradicting part (a), so $5 + \sqrt{5}$ is irrational **A1**

(b) Answer: $5 + \sqrt{5}$ is irrational

Question 23

(a) lets the integers be n and n + 1 and forms $n^2 + (n + 1)^2$ **M1**

(a) expands and simplifies to $2n^2 + 2n + 1$ **M1**

(a) writes this as $2(n^2 + n) + 1$, which is 2 x (an integer) + 1 **A1**

(a) concludes that it is odd for every integer n (cso) **A1**

(a) Answer: $n^2 + (n+1)^2 = 2(n^2 + n) + 1$, which is odd

(b) evaluates $5^2 + 6^2 = 25 + 36$ **M1**

(b) 61, which is odd (2 x 30 + 1) **A1**

(b) Answer: 61 is odd

Question 24

(a) pairs the factors: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$ **M1**

(a) lets $w = n^2 + 3n$ and writes the expression as $w(w + 2) + 1$ **M1**

(a) $w^2 + 2w + 1 = (w + 1)^2$ **A1**

(a) states $(n^2 + 3n + 1)^2$ with $n^2 + 3n + 1$ an integer (cso) **A1**

(a) Answer: $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$

(b) identifies $n = 2$ **M1**

(b) 11^2 **A1**

(b) Answer: 11^2

Question 25

(a) takes out the factor 2 **M1**

(a) $a = 2$, $p = -1$ **A1**

(a) $q = 1$ **A1**

(a) Answer: $f(x) = 2(x - 1)^2 + 1$

(b) states that a squared term is non-negative **M1**

(b) so $f(x) \geq 1 > 0$ (cso) **A1**

(b) Answer: minimum value $1 > 0$

(c) the minimum value 1 is above the x-axis, so there are no real roots **B1**

(c) Answer: (1, 1)

Question 26

(a) assumes $\sqrt{3} = a/b$ where a and b are integers with no common factor **M1**

(a) squares to obtain $a^2 = 3b^2$, so a^2 is a multiple of 3 **M1**

(a) deduces that a is a multiple of 3, $a = 3c$, and substitutes to obtain $3b^2 = 9c^2$, so $b^2 = 3c^2$ **A1**

(a) hence b is also a multiple of 3, contradicting that a and b have no common factor, so $\sqrt{3}$ is irrational (cso) **A1**

(a) Answer: $\sqrt{3}$ is irrational

(b) assumes $3 + \sqrt{3} = r$ with r rational **M1**

(b) then $\sqrt{3} = r - 3$ is rational, contradicting part (a), so $3 + \sqrt{3}$ is irrational **A1**

(b) Answer: $3 + \sqrt{3}$ is irrational

Question 27

(a) pairs the factors: $n(n + 3) = n^2 + 3n$ and $(n + 1)(n + 2) = n^2 + 3n + 2$ **M1**

(a) lets $w = n^2 + 3n$ and writes the expression as $w(w + 2) + 1$ **M1**

(a) $w^2 + 2w + 1 = (w + 1)^2$ **A1**

(a) states $(n^2 + 3n + 1)^2$ with $n^2 + 3n + 1$ an integer (cso) **A1**

(a) Answer: $n(n+1)(n+2)(n+3) + 1 = (n^2 + 3n + 1)^2$

(b) identifies $n = 4$ **M1**

(b) 29^2 **A1**

(b) Answer: 29^2

Question 28

(a) expands $\sin(\theta+60) = \sin(\theta)\cos60 + \cos(\theta)\sin60$ **M1**

(a) expands $\sin(\theta-60) = \sin(\theta)\cos60 - \cos(\theta)\sin60$ **M1**

(a) adds the two expansions so the $\cos(\theta)\sin60$ terms cancel, leaving $2\sin(\theta)\cos60$ **A1**

(a) substitutes the exact value $\cos60=1/2$ to obtain $2\sin(\theta)x(1/2)=\sin(\theta)$, cso **A1**

(a) Answer: Proof: $\sin(\theta+60)+\sin(\theta-60) = \sin(\theta)$

(b) uses the identity from part (a) to rewrite the equation as $\sin(\theta)=0.5$, ft **M1**

(b) $\theta=30$ **A1**

(b) $\theta=150$, cao, both values and no extras in range **A1**

(b) Answer: $\theta = 30$ or $\theta = 150$

Question 29

(a) states $(x-y)^2 \geq 0$ for all real x and y, since a square cannot be negative **B1**

(a) expands $(x-y)^2$ to $x^2-2xy+y^2$ **M1**

(a) rearranges $x^2-2xy+y^2 \geq 0$ to $x^2+y^2 \geq 2xy$, cso **A1**

(a) Answer: Proof: since $(x-y)^2 \geq 0$, $x^2+y^2 \geq 2xy$

(b) substitutes x=a, y=b into part (a) to obtain $a^2+b^2 \geq 2ab$, ft **M1**

(b) states ab>0 since a>0 and b>0, so dividing both sides of the inequality by ab does not change its direction **B1**

(b) divides both sides by ab to obtain $a/b+b/a \geq 2$, cso **A1**

(b) Answer: Proof: dividing $a^2+b^2 \geq 2ab$ by ab>0 gives $a/b+b/a \geq 2$

Question 30

(a) differentiates term by term **M1**

(a) $f'(x)=3x^2+12x+15$, cao **A1**

(a) Answer: $f'(x) = 3x^2 + 12x + 15$

(b) takes out the factor of 3: $f'(x)=3(x^2+4x+5)$ **M1**

(b) completes the square: $x^2+4x+5=(x+2)^2+1$, giving $f'(x)=3[(x+2)^2+1]$ **A1**

(b) states $(x+2)^2 \geq 0$ for all real x, so $(x+2)^2+1 \geq 1$, hence $f'(x) \geq 3 > 0$ for all real x, cso **A1**

(b) Answer: $f'(x) = 3[(x+2)^2+1] \geq 3 > 0$ for all real x

(c) states that since $f'(x)>0$ for all real x, f is an increasing function for all real values of x, cso, ft **B1**

(c) Answer: f(x) is an increasing function for all real x

Question 31

states $(x - 2y)^2 \geq 0$ for all real x and y, since a square cannot be negative **B1**

expands $(x - 2y)^2$ to $x^2 - 4xy + 4y^2$ **M1**

rearranges $x^2 - 4xy + 4y^2 \geq 0$ to $x^2 + 4y^2 \geq 4xy$, cso **A1**

Answer: Proof: since $(x-2y)^2 \geq 0$, $x^2 + 4y^2 \geq 4xy$

Question 32

(a) assumes n is odd, writes n=2k+1 for integer k, and expands $n^2=(2k+1)^2=4k^2+4k+1$ **M1**

(a) writes $n^2=2(2k^2+2k)+1$, which is odd; since "n odd implies n^2 odd" has been shown, its contrapositive "n^2 even implies n even" must also be true, cso **A1**

(a) Answer: Proof: if n^2 is even then n is even

(b) assumes, for contradiction, that $\sqrt{2}$ is rational, so $\sqrt{2}=p/q$ where p and q are integers with no common factor (q not equal 0), i.e. p/q is in its lowest terms **B1**

(b) squares both sides to obtain $p^2=2q^2$ **M1**

(b) deduces p^2 is even, so by part (a), p is even; writes p=2m for integer m, ft **A1**

(b) substitutes p=2m into $p^2=2q^2$ to get $4m^2=2q^2$, so $q^2=2m^2$, dependent on the previous M1 **dM1**

(b) deduces q^2 is even, so by part (a), q is even, ft **A1**

(b) states that p and q are both even, so they share a common factor of 2, contradicting the assumption that p/q has no common factor **B1**

(b) concludes that the original assumption is false, hence $\sqrt{2}$ is irrational, cso **B1**

(b) Answer: Proof by contradiction: $\sqrt{2}$ is irrational

Question 33

(a) takes out a factor of n: $n^3-n=n(n^2-1)$ **M1**

(a) factorises the difference of squares to give n(n-1)(n+1), cao **A1**

(a) Answer: $n^3-n = (n-1)n(n+1)$

(b) states that n and n+1 are consecutive integers, so exactly one of them is even, oe **B1**

(b) states that since the product (n-1)n(n+1) contains an even factor, the whole product is divisible by 2, cso **B1**

(b) Answer: Divisible by 2 because n and n+1 are consecutive, so one is even

(c) states that n-1, n and n+1 are three consecutive integers, and among any three consecutive integers exactly one is a multiple of 3, oe **B1**

(c) states that since the product contains a multiple of 3, the whole product is divisible by 3, cso **B1**

(c) Answer: Divisible by 3 because three consecutive integers include a multiple of 3

(d) states that 2 and 3 share no common factor other than 1, ft from (b) and (c) **M1**

(d) since (n-1)n(n+1) is divisible by both 2 and 3, it is divisible by 2x3=6 for every positive integer n, cso **A1**

(d) Answer: Proof complete: n^3-n is divisible by 6 for every positive integer n

Question 34

(a) states, for contradiction, that there are only finitely many primes, p_1, p_2, ..., p_k, and that this is a complete list of every prime, oe **B1**

(a) Answer: Assume there is a complete finite list of primes p_1, p_2, ..., p_k

(b) states that each p_i divides the product $p_1 \times p_2 \times ... \times p_k$ exactly, so N leaves remainder 1 when divided by any p_i, oe **B1**

(b) states that this holds for every i from 1 to k, so no prime in the list divides N, cso **B1**

(b) Answer: N leaves remainder 1 on division by each p_i, so no p_i divides N

(c) states that N>1, so N must have at least one prime factor, since every integer greater than 1 has a prime factor **B1**

(c) states that this prime factor cannot be any of p_1, ..., p_k (by part (b)), so it is a prime missing from the original list, contradicting the assumption that the list was complete, cso, ft **B1**

(c) Answer: N has a prime factor not in the original list, contradicting completeness of the list

(d) states that the assumption of finitely many primes is false, so there are infinitely many primes, cso, ft **B1**

(d) Answer: There are infinitely many primes

(e) calculates N=2x3x5+1=31 **M1**

(e) verifies 31 is not divisible by 2, 3 or 5 (e.g. remainder 1 in each case) **A1**

(e) identifies that 31 is itself prime, a prime not in the original list {2, 3, 5}, cao **A1**

(e) Answer: N=31, which is prime and not in {2, 3, 5}

(f) calculates N=2x3x5x7x11x13+1=30031, and notes 30031=59x509 is composite **B1**

(f) states that neither 59 nor 509 is in the original list {2,3,5,7,11,13}, so N still has a prime factor missing from the list, confirming the argument holds whether or not N is prime, cso **B1**

(f) Answer: N=30031=59x509; both 59 and 509 are new primes not in the original list