1.9 Pure: Numerical Methods: 1-hour Pack

EDEXCEL 9MA0 · Calculator allowed · about 60 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 The table shows values of $y = \ln(x + 3)$, rounded to 4 decimal places, for equally spaced values of x.

x	0	0.75	1.5	2.25	3
y	1.0986	1.3218	1.5041	1.6582	1.7918

(a) Use the trapezium rule with 4 strips to estimate the integral from $x = 0$ to $x = 3$ of $\ln(x + 3)\,dx$, giving your answer to 4 decimal places. **(4)**

(b) State, with a reason, whether the trapezium rule gives an overestimate or an underestimate. **(2)**

(c) Find the percentage error in your estimate, given that the exact value is 4.4547, to 2 significant figures. **(2)**

(Total for Question 1 is 8 marks)

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2 $f(x) = x^3 + 2x - 5$. The equation $f(x) = 0$ has a root α near $x = 2$.

(a) Find $f'(x)$. **(2)**

(b) Use the Newton-Raphson method with $x_0 = 2$ to find x_1, x_2 and x_3, giving each answer to 4 decimal places. **(4)**

(c) Hence state the value of α correct to 3 decimal places, and justify your answer by evaluating f at two suitable values. **(3)**

(Total for Question 2 is 9 marks)

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3 This question tests your understanding of key ideas used in numerical methods.

(a) State the condition on f(x) that is required for the change-of-sign (sign change) test for a root to be valid on an interval [a,b]. **(1)**

(b) Write down the general iteration formula used in the Newton-Raphson method for estimating a root of f(x) = 0. **(1)**

(c) The function f(x) = 1/(x-2) satisfies f(1) < 0 and f(3) > 0. Explain why the change-of-sign method alone does not guarantee a root of f(x) = 0 between x = 1 and x = 3. **(2)**

(d) Give one reason why the Newton-Raphson method may fail to converge to a root if the starting value x_0 is chosen close to a stationary point of y = f(x). **(2)**

(e) The iteration $x_{n+1} = g(x_n)$ converges to a root α. State, with a reason, the condition on g'(α) required for this convergence. **(2)**

(Total for Question 3 is 8 marks)

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4 Let $f(x) = x^3 - 7x + 3$.

(a) Show that f(x) = 0 has a root between x = 0 and x = 1. **(2)**

(b) Show that, using the Newton-Raphson method with $x_0 = 0.5$, $x_1 = 0.44$. **(2)**

(c) Find x_2 and x_3 using the Newton-Raphson method, giving each answer to 6 decimal places, and hence state the root of f(x) = 0 correct to 4 decimal places. **(3)**

(d) Show that the root found in part (c) is x = 0.441 correct to 3 decimal places. **(2)**

(Total for Question 4 is 9 marks)

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5 The curve $y = 1/x$ is defined for $1 \leq x \leq 3$. The table gives values of y.

x : 1 1.5 2 2.5 3

y : 1 ? ? 0.4 0.3333

(a) Find the missing values of y at $x = 1.5$ and $x = 2$, each to 4 decimal places. **(2)**

(b) Use the trapezium rule with all 4 strips to find an estimate for the integral of $1/x$ from 1 to 3, giving your answer to 4 decimal places. **(3)**

(c) Given that the exact value of the integral of $1/x$ from 1 to 3 is $\ln(3)$, calculate the percentage error in the estimate found in part (b), giving your answer to 2 decimal places. **(3)**

(d) By considering the second derivative of $y = 1/x$, explain why the trapezium rule overestimates the integral on this interval. **(2)**

(Total for Question 5 is 10 marks)

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6 Let $f(x) = x^3 - 6x + 2$. Evaluate f(3).

(Total for Question 6 is 1 mark)

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7 Show that the equation $x^3 - 6x + 2 = 0$ has a root between $x = 2$ and $x = 3$.

(Total for Question 7 is 2 marks)

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8 The equation $x^3 - 2x - 5 = 0$ has a root α near $x = 2$. Two rearrangements are suggested: $x_{n+1} = (2x_n + 5)^{1/3}$ and $x_{n+1} = (x_n^3 - 5)/2$. Using $x_0 = 2$ for each, find x_1 and x_2 for both iterations, giving each value to 4 decimal places, and hence state, with a reason, which rearrangement converges to α.

(Total for Question 8 is 4 marks)

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Mark scheme · 1.9 Pure: Numerical Methods: 1-hour Pack

Question 1

(a) h = 3/4 = 0.75 **B1**

(a) trapezium rule: (h/2)[y_0 + y_n + 2(sum of the middle values)] **M1**

(a) sum of middle values = 4.4841 **A1**

(a) 4.4470 **A1**

(a) Answer: 4.4470

(b) underestimate **B1**

(b) the curve is concave (concave down), so the chords lie below the curve **B1**

(b) Answer: underestimate

(c) |estimate - exact|/exact x 100 **M1**

(c) 0.17% **A1**

(c) Answer: 0.17%

Question 2

(a) differentiates each term **M1**

(a) $f'(x) = 3x^2 + 2$ **A1**

(a) Answer: $f'(x) = 3x^2 + 2$

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **M1**

(b) $x_1 = 1.5000$ **A1**

(b) $x_2 = 1.3429$ **A1**

(b) $x_3 = 1.3284$ **A1**

(b) Answer: $x_1 = 1.5000$, $x_2 = 1.3429$, $x_3 = 1.3284$

(c) α = 1.328 **B1**

(c) evaluates f(1.3275) and f(1.3285) **M1**

(c) values -0.0056 and 0.0017 have opposite signs **A1**

(c) Answer: α = 1.328

Question 3

(a) f must be continuous on [a,b] **B1**

(a) Answer: f(x) must be continuous on the interval [a,b].

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **B1**

(b) Answer: $x_{n+1} = x_n - f(x_n)/f'(x_n)$

(c) f(x) = 1/(x-2) is discontinuous (has a vertical asymptote) at x = 2, which lies inside the interval [1,3] **B1**

(c) the change of sign is caused by the discontinuity, not by a root; f(x) = 0 has no solution at all here, so the method's condition (continuity) is not met, and its conclusion is invalid cso **B1**

(c) Answer: f is discontinuous at x=2 in the interval, so the sign change is due to the asymptote, not a root; the method requires continuity.

(d) near a stationary point f'(x) is close to zero, so the tangent to the curve is nearly horizontal **B1**

(d) the tangent then meets the x-axis a long way from x_0 (or, if $f'(x_0)=0$ exactly, the formula requires division by zero and cannot be used), so the next iterate can be far from the required root **B1**

(d) Answer: $f'(x_0)$ is close to (or equal to) zero, so the near-horizontal tangent gives a next estimate far from the root, or the formula is undefined.

(e) $|g'(\alpha)| < 1$ **B1**

(e) reason: this ensures the size of the error reduces at each step, so successive iterates get closer to α **B1**

(e) Answer: $|g'(\alpha)| < 1$, so that the error decreases at each iteration.

Question 4

(a) evaluates f(0) and f(1) **M1**

(a) f(0) = 3, f(1) = -3; opposite signs and f continuous, so by the change of sign rule a root lies in (0,1) cso **A1**

(a) Answer: f(0) = 3, f(1) = -3, sign change confirms a root in (0,1)

(b) states $f'(x) = 3x^2 - 7$ and substitutes $x_0 = 0.5$ into $x_1 = x_0 - f(x_0)/f'(x_0)$, using f(0.5) = -0.375 and f'(0.5) = -6.25 **M1**

(b) $x_1 = 0.5 - (-0.375/-6.25) = 0.44$ cso **A1**

(b) Answer: $x_1 = 0.44$

(c) correct Newton-Raphson substitution using $x_1 = 0.44$ **M1**

(c) $x_2 = 0.440808$ (awrt 6dp) **A1**

(c) root = 0.4408 (4dp) cao ($x_3 = 0.440808$, confirming stability) **A1**

(c) Answer: $x_2 = 0.440808$, root = 0.4408 (4dp)

(d) evaluates f(0.4405) and f(0.4415) **M1**

(d) f(0.4405) = +0.0020 (>0) and f(0.4415) = -0.0044 (<0); change of sign confirms root = 0.441 to 3dp cso **A1**

(d) Answer: root = 0.441 (3dp)

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Question 5

(a) y(1.5) = 0.6667 (awrt 4dp) **B1**

(a) y(2) = 0.5 **B1**

(a) Answer: y(1.5) = 0.6667, y(2) = 0.5

(b) correct trapezium structure with h = 0.5: (h/2) $[y_0+y_4+2(y_1+y_2+y_3)]$ **M1**

(b) correct value inside the bracket, 4.4667 **A1**

(b) 1.1167 (awrt 4dp) **A1**

(b) Answer: Estimate = 1.1167

(c) attempts percentage error = |estimate - exact|/exact x 100 **M1**

(c) uses ln(3) = 1.0986 (awrt 4dp) **A1**

(c) 1.64% (awrt 2dp) **A1**

(c) Answer: Percentage error = 1.64%

(d) $d2y/dx2 = 2/x^3 > 0$ for $1 \leq x \leq 3$, so the curve is convex on this interval **B1**

(d) hence the chords lie above the curve, so the trapezium rule overestimates the true area **B1**

(d) Answer: y = 1/x is convex on [1,3] (d2y/dx2 > 0), so chords lie above the curve, giving an overestimate.

Question 6

11 cao **B1**

Answer: 11

Question 7

evaluates f(2) = -2 and f(3) = 11 **M1**

states f(2) < 0 and f(3) > 0, a change of sign, so a root lies between x=2 and x=3 **A1**

Answer: f(2) = -2 and f(3) = 11, a sign change, so a root lies between x=2 and x=3.

Question 8

finds x_1 = awrt 2.0801 and x_2 = awrt 2.0924 for $x_{n+1} = (2x_n+5)^{1/3}$ **M1**

finds $x_1 = 1.5000$ and $x_2 = -0.8125$ for $x_{n+1} = (x_n^3-5)/2$ **A1**

compares the two sequences: the first is settling towards a value near 2.09, the second is moving further away from x_0 **M1**

concludes $x_{n+1} = (2x_n+5)^{1/3}$ converges to α (the values are approaching a fixed value), while $x_{n+1} = (x_n^3-5)/2$ diverges (the values move away from the root) **A1**

Answer: First iteration converges to α (approximately 2.09); second iteration diverges

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