1.9 Pure: Numerical Methods: 2-hour Pack

EDEXCEL 9MA0 · Calculator allowed · about 120 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 $f(x) = \ln x + x - 1$. The equation $f(x) = 0$ has a root α near $x = 3$.

(a) Find $f'(x)$. **(2)**

(b) Use the Newton-Raphson method with $x_0 = 3$ to find x_1, x_2 and x_3, giving each answer to 4 decimal places. **(4)**

(c) Hence state the value of α correct to 3 decimal places, and justify your answer by evaluating f at two suitable values. **(3)**

(Total for Question 1 is 9 marks)

2 Let $f(x) = \ln(x) + x - 4$, for $x > 0$.

(a) Show that $f(x) = 0$ has a root between $x = 2$ and $x = 3$. **(2)**

(b) Show that $f'(x) = 1/x + 1$, and hence write down the Newton-Raphson iteration formula for solving $f(x) = 0$. **(2)**

(c) Using $x_0 = 3$, find x_1 and x_2 using the Newton-Raphson method, giving each answer to 6 decimal places. **(3)**

(d) Prove that the root of $f(x) = 0$ is $x = 2.93$ correct to 2 decimal places. **(2)**

(Total for Question 2 is 9 marks)

3 For the iteration $x_{n+1} = g(x_n)$ converging to a root α, state the condition on $g'(x)$ near α.

(Total for Question 3 is 1 mark)

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4 Let $f(x) = x^3 - 3x + 1$.

(a) Show that $f'(x) = 0$ when $x = 1$, and explain why the Newton-Raphson formula cannot be used with $x_0 = 1$ to estimate a root of $f(x) = 0$. **(2)**

(b) Show that $f(x) = 0$ has a root between $x = 1.4$ and $x = 1.6$. **(2)**

(c) Using $x_0 = 1.5$, apply the Newton-Raphson method to find this root correct to 3 decimal places, showing your iterations. **(4)**

(d) Prove that this root is correct to 3 decimal places by considering a suitable sign change. **(2)**

(Total for Question 4 is 10 marks)

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5 The equation $x^2 + 8x - 10 = 0$ has a positive root, α. The iteration $x_{n+1} = (10 - x_n^2)/8$ is used to estimate α, starting from $x_0 = 1$.

(a) Use the quadratic formula to show that the exact positive root of $x^2 + 8x - 10 = 0$ is $\alpha = -4 + \sqrt{26}$. **(3)**

(b) Show that $x^2 + 8x - 10 = 0$ can be rearranged to give $x = (10 - x^2)/8$. **(1)**

(c) Find $g'(x)$, and show that $|g'(\alpha)| < 1$. **(2)**

(d) By finding x_1, x_2, x_3 and x_4 (each to 4dp), show that the iteration converges to α by oscillating alternately above and below the true value (a cobweb pattern), rather than approaching α from one side only (a staircase pattern). **(3)**

(e) State how the resulting diagram would differ if $g'(\alpha)$ were positive instead of negative, with $|g'(\alpha)|$ still less than 1. **(1)**

(Total for Question 5 is 10 marks)

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6 Let $f(x) = x^3 - x - 3$.

(a) Show that $f(x) = 0$ has a root between $x = 1$ and $x = 2$. **(2)**

(b) Using the iteration $x_{n+1} = (x_n + 3)^{1/3}$ with $x_0 = 1.5$, find x_1, x_2, x_3 and x_4, giving each to 4 decimal places, and state the root correct to 4 decimal places. **(4)**

(c) Show that the alternative rearrangement $x_{n+1} = x_n^3 - 3$, using the same $x_0 = 1.5$, does not converge to the root found in part (b) by finding x_1 and x_2. **(3)**

(d) Use the Newton-Raphson method with $x_0 = 1.7$ to find the root of $f(x) = 0$ correct to 4 decimal places, showing at least two iterations. **(4)**

(e) By comparing your answers to parts (b) and (d), state which method reached the root to 4 decimal places using fewer iterations. **(1)**

(Total for Question 6 is 14 marks)

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7 Using $x_0 = 1$ and the iteration $x_{n+1} = (2x_n + 5)^{1/2}$, find x_1 and x_2, each to 4 decimal places.

(Total for Question 7 is 2 marks)

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8 Let $f(x) = x^3 - 6x + 2$. Find $f'(x)$, and hence find $f'(2)$.

(Total for Question 8 is 2 marks)

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9 Let $f(x) = x^3 - 6x + 2$, with $f'(x) = 3x^2 - 6$. Using $x_0 = 2$, apply the Newton-Raphson method to find x_1, giving your answer to 4 decimal places.

(Total for Question 9 is 2 marks)

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10 The table shows values of $y = 1/(x+1)$ for $x = 0, 1, 2, 3, 4$. x: 0, 1, 2, 3, 4 y: 1, 0.5, 0.3333, 0.25, 0.2. Use the trapezium rule with all four strips (h=1) to find an estimate for the integral of $1/(x+1)$ from x=0 to x=4, giving your answer to 4 decimal places.

(Total for Question 10 is 2 marks)

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11 Show that the equation $x^3 + 3x - 7 = 0$ has a root between $x = 1$ and $x = 2$. Show that this equation can be rearranged into the iterative form $x_{n+1} = (7 - 3x_n)^{1/3}$.

(Total for Question 11 is 3 marks)

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12 Using $x_0 = 1.5$, apply the iteration $x_{n+1} = (7 - 3x_n)^{1/3}$ to find x_1, x_2 and x_3, giving each value to 4 decimal places. Hence write down an estimate for the root of $x^3 + 3x - 7 = 0$, correct to 1 decimal place.

(Total for Question 12 is 3 marks)

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13 Using $x_0 = 2$ and the Newton-Raphson iteration $x_{n+1} = x_n - (x_n^2 - 7)/(2x_n)$, find x_1, giving your exact answer.

(Total for Question 13 is 1 mark)

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14 Using $x_0 = 1$ and the iteration $x_{n+1} = (2x_n + 5)^{1/2}$, find x_1, giving your answer to 4 decimal places.

(Total for Question 14 is 1 mark)

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15 Let $f(x) = 2\sin(x) - x + 1$, where x is measured in radians.

(a) Show that $f(x) = 0$ has a root between $x = 2.3$ and $x = 2.4$. **(2)**

(b) Find $f'(x)$, and hence write down the Newton-Raphson iteration formula for solving $f(x) = 0$. **(2)**

(c) Using $x_0 = 2.3$, find x_1 and x_2, giving each answer to 4 decimal places, and state the root of $f(x) = 0$ correct to 4 decimal places. **(4)**

(d) Show that this root is $x = 2.38$ correct to 2 decimal places. **(2)**

(Total for Question 15 is 10 marks)

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16 The curve $y = \ln(x+1)$ is defined for $0 \leq x \leq 4$. The table gives values of y at $x = 0, 1, 2, 3, 4$. x: 0, 1, 2, 3, 4 y: 0, 0.6931, p, 1.3863, q. Find the values of p and q, each correct to 4 decimal places. Use the trapezium rule with all four strips (h=1) to find an estimate for the integral of $\ln(x+1)$ from x=0 to x=4, giving your answer to 4 decimal places.

(Total for Question 16 is 4 marks)

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17 Let $f(x) = xe^x - 5$. This question compares two numerical methods for solving $f(x) = 0$.

(a) Show that $f(x) = 0$ has a root between $x = 1.3$ and $x = 1.4$. **(2)**

(b) Show that the equation $xe^x = 5$ can be rearranged into the iterative form $x_{n+1} = \ln(5/x_n)$. **(2)**

(c) Using $x_0 = 1.3$, find x_1, x_2, x_3 and x_4, giving each to 4 decimal places, and comment on the pattern of convergence. **(4)**

(d) By first finding $f'(x)$ using the product rule, use the Newton-Raphson method with $x_0 = 1.3$ to find x_1 and x_2, giving each to 6 decimal places, and state the root of $f(x) = 0$ correct to 4 decimal places. **(5)**

(e) Comment on the relative rates of convergence of the two methods used in parts (c) and (d), referring to the number of iterations required to obtain the root correct to 4 decimal places. **(2)**

(Total for Question 17 is 15 marks)

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Mark scheme · 1.9 Pure: Numerical Methods: 2-hour Pack

Question 1

(a) differentiates each term **M1**

(a) f'(x) = 1/x + 1 **A1**

(a) Answer: f'(x) = 1/x + 1

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **M1**

(b) $x_1 = 0.6760$ **A1**

(b) $x_2 = 0.9646$ **A1**

(b) $x_3 = 0.9997$ **A1**

(b) Answer: $x_1 = 0.6760$, $x_2 = 0.9646$, $x_3 = 0.9997$

(c) α = 1.000 **B1**

(c) evaluates f(0.9995) and f(1.0005) **M1**

(c) values -0.0010 and 0.0010 have opposite signs **A1**

(c) Answer: α = 1.000

Question 2

(a) evaluates f(2) and f(3) **M1**

(a) f(2) = -1.3069 (<0), f(3) = 0.0986 (>0); opposite signs and f continuous on [2,3], so a root lies in (2,3) cso **A1**

(a) Answer: f(2) = -1.3069, f(3) = 0.0986, sign change confirms a root in (2,3)

(b) f'(x) = 1/x + 1, obtained by differentiating ln(x) + x - 4 term by term cso **B1**

(b) $x_{n+1} = x_n - (\ln(x_n)+x_n-4)/(1/x_n + 1)$ **B1**

(b) Answer: f'(x) = 1/x + 1; $x_{n+1} = x_n - (\ln(x_n)+x_n-4)/(1/x_n+1)$

(c) correct Newton-Raphson substitution with $x_0 = 3$ **M1**

(c) $x_1 = 2.926041$ (awrt 6dp) **A1**

(c) $x_2 = 2.926271$ (awrt 6dp) **A1**

(c) Answer: $x_1 = 2.926041$, $x_2 = 2.926271$

(d) evaluates f(2.925) and f(2.935) **M1**

(d) f(2.925) = -0.0017 (<0) and f(2.935) = +0.0117 (>0); change of sign confirms root = 2.93 to 2dp cso **A1**

(d) Answer: root = 2.93 (2dp)

Question 3

|g'(α)| < 1 **B1**

Answer: |g'(α)| < 1

Question 4

(a) $f'(x) = 3x^2 - 3$, so f'(1) = 3-3 = 0 cso **M1**

(a) the Newton-Raphson formula requires dividing by $f'(x_0)$; since f'(1) = 0 this division is undefined, so x_1 cannot be found (geometrically, the tangent at x=1 is horizontal and never meets the x-axis) **A1**

(a) Answer: f'(1) = 0, so the formula requires division by zero and cannot be used.

(b) evaluates f(1.4) and f(1.6) **M1**

(b) f(1.4) = -0.456 (<0), f(1.6) = 0.296 (>0); sign change confirms root in (1.4,1.6) cso **A1**

(b) Answer: f(1.4) = -0.456, f(1.6) = 0.296, sign change confirms a root in (1.4,1.6)

(c) correct Newton-Raphson substitution with $x_0 = 1.5$, using $f'(x) = 3x^2-3$ **M1**

(c) $x_1 = 1.5333$ (awrt 4dp) **A1**

(c) $x_2 = 1.5321$ (awrt 4dp) **A1**

(c) root = 1.532 (3dp) cao **A1**

(c) Answer: $x_1 = 1.533333$, $x_2 = 1.532091$, root = 1.532 (3dp)

(d) evaluates f(1.5315) and f(1.5325) **M1**

(d) f(1.5315) = -0.0024 (<0), f(1.5325) = +0.0017 (>0); sign change confirms root = 1.532 to 3dp cso **A1**

(d) Answer: root = 1.532 (3dp)

Question 5

(a) correct application of the quadratic formula with a=1, b=8, c=-10 **M1**

(a) $x = (-8 \pm \sqrt{64+40})/2 = -4 \pm \sqrt{26}$ **A1**

(a) takes the positive root, $\alpha = -4 + \sqrt{26}$ cso **A1**

(a) Answer: $\alpha = -4 + \sqrt{26}$

(b) correct rearrangement: $8x = 10 - x^2$, so $x = (10-x^2)/8$ cso **B1**

(b) Answer: $x = (10-x^2)/8$

(c) g'(x) = -2x/8 = -x/4 **M1**

(c) g'(α) = -1.0990/4 = -0.2748, so |g'(α)| = 0.2748 < 1 cso **A1**

(c) Answer: g'(x) = -x/4; |g'(α)| = 0.2748 < 1

(d) correct repeated substitution into the iteration formula **M1**

(d) $x_1 = 1.1250$, $x_2 = 1.0918$, $x_3 = 1.1010$, $x_4 = 1.0985$ (awrt 4dp) **A1**

(d) values alternate above (1.1250, 1.1010) and below (1.0918, 1.0985) α = 1.0990, consistent with g'(α) < 0, giving a cobweb (not staircase) diagram cso **A1**

(d) Answer: x_1=1.1250, x_2=1.0918, x_3=1.1010, x_4=1.0985; values oscillate either side of α, confirming a cobweb pattern.

(e) the iteration would converge monotonically from one side only (a staircase diagram) rather than oscillating **B1**

(e) Answer: It would give a staircase diagram, converging from one side only.

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Question 6

(a) evaluates f(1) and f(2) **M1**

(a) f(1) = -3, f(2) = 3; opposite signs and f continuous on [1,2], so a root lies in (1,2) cso **A1**

(a) Answer: f(1) = -3, f(2) = 3, sign change confirms a root in (1,2)

(b) correct repeated substitution into the iteration formula **M1**

(b) $x_1 = 1.6510$ (awrt 4dp) **A1**

(b) $x_2 = 1.6692$, $x_3 = 1.6714$ (awrt 4dp each) **A1**

(b) $x_4 = 1.6717$ and hence root = 1.6717 (4dp) cao **A1**

(b) Answer: x_1=1.6510, x_2=1.6692, x_3=1.6714, x_4=1.6717; root = 1.6717 (4dp)

(c) correct substitution $x_1 = 1.5^3 - 3$ **M1**

(c) $x_1 = 0.375$, $x_2 = 0.375^3 - 3 = -2.9473$ (awrt 4dp) **A1**

(c) the values are moving rapidly away from 1.6717, so this rearrangement diverges rather than converging cso **A1**

(c) Answer: $x_1 = 0.375$, $x_2 = -2.9473$; values diverge away from the root.

(d) states $f'(x) = 3x^2 - 1$ **M1**

(d) correct Newton-Raphson substitution with $x_0 = 1.7$ **M1**

(d) $x_1 = 1.6722$ (awrt 4dp) **A1**

(d) $x_2 = 1.6717$ and root = 1.6717 (4dp) cao **A1**

(d) Answer: $x_1 = 1.672229$, $x_2 = 1.671700$, root = 1.6717 (4dp)

(e) the Newton-Raphson method (part (d)), which reached 1.6717 after 2 iterations, compared with 4 iterations needed for the rearrangement in part (b) **B1**

(e) Answer: Newton-Raphson converged faster (2 iterations versus 4).

Question 7

finds $x_1 = 2.6458$ (or uses the unrounded value) **M1**

x_2 = awrt 3.2080 **A1**

Answer: $x_1 = 2.6458$, $x_2 = 3.2080$

Question 8

differentiates to find $f'(x) = 3x^2 - 6$ **M1**

6 cao **A1**

Answer: $f'(x) = 3x^2 - 6$; $f'(2) = 6$

Question 9

substitutes into $x_1 = x_0 - f(x_0)/f'(x_0) = 2 - (-2)/6$ **M1**

awrt 2.3333 **A1**

Answer: $x_1 = 2.3333$

Question 10

substitutes into the trapezium rule, $(1/2)[(y_0+y_4) + 2(y_1+y_2+y_3)]$ **M1**

awrt 1.6833 **A1**

Answer: 1.6833

Question 11

evaluates f(1) = -3 and f(2) = 7, a change of sign, so a root lies between x=1 and x=2 **M1**

rearranges $x^3 + 3x - 7 = 0$ to $x^3 = 7 - 3x$ **M1**

takes the cube root of both sides to give $x = (7-3x)^{1/3}$ (shown) **A1**

Answer: Root between x=1 and x=2 (shown); rearrangement $x_{n+1} = (7-3x_n)^{1/3}$ (shown)

Question 12

finds x_1 = awrt 1.3572 **M1**

finds x_2 = awrt 1.4307 and x_3 = awrt 1.3938 **A1**

1.4 cao (ft from x_3, or from the known trend of the iteration) **A1**

Answer: $x_1 = 1.3572$, $x_2 = 1.4307$, $x_3 = 1.3938$; root is approximately 1.4

Question 13

2.75 cao, oe 11/4 **B1**

Answer: $x_1 = 2.75$

Question 14

awrt 2.6458 **B1**

Answer: $x_1 = 2.6458$

Question 15

(a) evaluates f(2.3) and f(2.4) in radians **M1**

(a) f(2.3) = +0.1914 (>0), f(2.4) = -0.0491 (<0); sign change confirms root in (2.3,2.4) cso **A1**

(a) Answer: f(2.3) = 0.1914, f(2.4) = -0.0491, sign change confirms a root in (2.3,2.4)

(b) $f'(x) = 2\cos(x) - 1$ **B1**

(b) $x_{n+1} = x_n - (2\sin(x_n)-x_n+1)/(2\cos(x_n)-1)$ **B1**

(b) Answer: $f'(x) = 2\cos(x) - 1$; $x_{n+1} = x_n - (2\sin(x_n)-x_n+1)/(2\cos(x_n)-1)$

(c) correct Newton-Raphson substitution with $x_0 = 2.3$ **M1**

(c) $x_1 = 2.3821$ (awrt 4dp) **A1**

(c) $x_2 = 2.3801$ (awrt 4dp) **A1**

(c) root = 2.3801 (4dp) cao **A1**

(c) Answer: $x_1 = 2.3821$, $x_2 = 2.3801$, root = 2.3801 (4dp)

(d) evaluates f(2.375) and f(2.385) **M1**

(d) f(2.375) = +0.0124 (>0), f(2.385) = -0.0121 (<0); sign change confirms root = 2.38 to 2dp cso **A1**

(d) Answer: root = 2.38 (2dp)

Question 16

p = ln(3) = awrt 1.0986 **B1**

q = ln(5) = awrt 1.6094 **B1**

substitutes into the trapezium rule, $(1/2)[(y_0+y_4) + 2(y_1+y_2+y_3)]$ **M1**

awrt 3.9828 **A1**

Answer: p = 1.0986, q = 1.6094; trapezium estimate = 3.9828

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Question 17

(a) evaluates f(1.3) and f(1.4) **M1**

(a) f(1.3) = -0.2299 (<0), f(1.4) = +0.6773 (>0); sign change confirms a root in (1.3,1.4) cso **A1**

(a) Answer: f(1.3) = -0.2299, f(1.4) = 0.6773, sign change confirms a root in (1.3,1.4)

(b) rearranges $xe^x = 5$ to $e^x = 5/x$ **M1**

(b) takes ln of both sides to give $x = \ln(5/x)$ cso **A1**

(b) Answer: $x_{n+1} = \ln(5/x_n)$

(c) correct repeated substitution into the iteration formula **M1**

(c) $x_1 = 1.3471$ (awrt 4dp) **A1**

(c) $x_2 = 1.3115$, $x_3 = 1.3383$ (awrt 4dp each) **A1**

(c) $x_4 = 1.3181$ (awrt 4dp), with a valid comment that the values oscillate above and below the root and are converging only slowly **A1**

(c) Answer: x_1=1.3471, x_2=1.3115, x_3=1.3383, x_4=1.3181; the iteration oscillates and converges slowly.

(d) $f'(x) = e^x + xe^x = e^x(1+x)$ (product rule) **B1**

(d) correct Newton-Raphson substitution with $x_0 = 1.3$ **M1**

(d) $x_1 = 1.327243$ (awrt 6dp) **A1**

(d) $x_2 = 1.326725$ (awrt 6dp) **A1**

(d) root = 1.3267 (4dp) cao **A1**

(d) Answer: $f'(x)=e^x(1+x)$; x_1=1.327243, x_2=1.326725, root = 1.3267 (4dp)

(e) the Newton-Raphson method converges much faster, giving the root to 4dp after only 2 iterations, whereas the iteration in (c) is still oscillating and has not settled to 4dp after 4 iterations **B1**

(e) valid supporting reason, e.g. Newton-Raphson uses gradient information to make a much closer next estimate at each step, so fewer iterations are needed for the same accuracy **B1**

(e) Answer: Newton-Raphson converges far more quickly (2 iterations vs. more than 4) because it uses the gradient of f to target the root directly.

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