1.9 Pure: Numerical Methods

EDEXCEL 9MA0 · Calculator allowed · about 230 minutes · Write-in edition

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 The table shows values of $y = \ln(x + 3)$, rounded to 4 decimal places, for equally spaced values of x.

x	0	0.8	1.6	2.4	3.2	4
y	1.0986	1.3350	1.5261	1.6864	1.8245	1.9459

(a) Use the trapezium rule with 5 strips to estimate the integral from $x = 0$ to $x = 4$ of $\ln(x + 3)\,dx$, giving your answer to 4 decimal places. **(4)**

(b) State, with a reason, whether the trapezium rule gives an overestimate or an underestimate. **(2)**

(c) Find the percentage error in your estimate, given that the exact value is 6.3255, to 2 significant figures. **(2)**

(Total for Question 1 is 8 marks)

2 $f(x) = x^3 - 3x - 9$, and the equation $f(x) = 0$ has a single real root, α.

(a) Show that α lies between $x = 2$ and $x = 3$. **(2)**

(b) Show that the equation can be rearranged as $x = (3x + 9)^{1/3}$. **(2)**

(c) Use $x_{n+1} = (3x_n + 9)^{1/3}$ with $x_0 = 2$ to find x_1, x_2 and x_3, giving each to 4 decimal places. **(3)**

(d) By considering the change of sign of f at $x = 2.545$ and $x = 2.555$, show that $\alpha = 2.55$ to 2 decimal places. **(2)**

(Total for Question 2 is 9 marks)

3 The table shows values of $y = \sqrt{x + 2}$, rounded to 4 decimal places, for equally spaced values of x.

x	0	0.4	0.8	1.2	1.6	2
y	1.4142	1.5492	1.6733	1.7889	1.8974	2.0000

(a) Use the trapezium rule with 5 strips to estimate the integral from $x = 0$ to $x = 2$ of $\sqrt{x + 2}\ dx$, giving your answer to 4 decimal places. **(4)**

(b) State, with a reason, whether the trapezium rule gives an overestimate or an underestimate. **(2)**

(c) Find the percentage error in your estimate, given that the exact value is 3.4477, to 2 significant figures. **(2)**

(Total for Question 3 is 8 marks)

4 The table shows values of $y = \ln(x + 3)$, rounded to 4 decimal places, for equally spaced values of x.

x	0	0.75	1.5	2.25	3
y	1.0986	1.3218	1.5041	1.6582	1.7918

(a) Use the trapezium rule with 4 strips to estimate the integral from $x = 0$ to $x = 3$ of $\ln(x + 3)\,dx$, giving your answer to 4 decimal places. **(4)**

(b) State, with a reason, whether the trapezium rule gives an overestimate or an underestimate. **(2)**

(c) Find the percentage error in your estimate, given that the exact value is 4.4547, to 2 significant figures. **(2)**

(Total for Question 4 is 8 marks)

5 $f(x) = x^3 + x - 7$. The equation $f(x) = 0$ has a root α near $x = 3$.

(a) Find $f'(x)$. **(2)**

(b) Use the Newton-Raphson method with $x_0 = 3$ to find x_1, x_2 and x_3, giving each answer to 4 decimal places. **(4)**

(c) Hence state the value of α correct to 3 decimal places, and justify your answer by evaluating f at two suitable values. **(3)**

(Total for Question 5 is 9 marks)

6 $f(x) = x^3 + 2x - 5$. The equation $f(x) = 0$ has a root α near $x = 2$.

(a) Find $f'(x)$. **(2)**

(b) Use the Newton-Raphson method with $x_0 = 2$ to find x_1, x_2 and x_3, giving each answer to 4 decimal places. **(4)**

(c) Hence state the value of α correct to 3 decimal places, and justify your answer by evaluating f at two suitable values. **(3)**

(Total for Question 6 is 9 marks)

7 $f(x) = \ln x + x - 1$. The equation $f(x) = 0$ has a root α near $x = 3$.

(a) Find $f'(x)$. **(2)**

(b) Use the Newton-Raphson method with $x_0 = 3$ to find x_1, x_2 and x_3, giving each answer to 4 decimal places. **(4)**

(c) Hence state the value of α correct to 3 decimal places, and justify your answer by evaluating f at two suitable values. **(3)**

(Total for Question 7 is 9 marks)

8 This question tests your understanding of key ideas used in numerical methods.

(a) State the condition on f(x) that is required for the change-of-sign (sign change) test for a root to be valid on an interval [a,b]. **(1)**

(b) Write down the general iteration formula used in the Newton-Raphson method for estimating a root of f(x) = 0. **(1)**

(c) The function f(x) = 1/(x-2) satisfies f(1) < 0 and f(3) > 0. Explain why the change-of-sign method alone does not guarantee a root of f(x) = 0 between x = 1 and x = 3. **(2)**

(d) Give one reason why the Newton-Raphson method may fail to converge to a root if the starting value x_0 is chosen close to a stationary point of y = f(x). **(2)**

(e) The iteration $x_{n+1} = g(x_n)$ converges to a root α. State, with a reason, the condition on g'(α) required for this convergence. **(2)**

(Total for Question 8 is 8 marks)

9 Let $f(x) = x^3 - 7x + 3$.

(a) Show that $f(x) = 0$ has a root between $x = 0$ and $x = 1$. **(2)**

(b) Show that, using the Newton-Raphson method with $x_0 = 0.5$, $x_1 = 0.44$. **(2)**

(c) Find x_2 and x_3 using the Newton-Raphson method, giving each answer to 6 decimal places, and hence state the root of $f(x) = 0$ correct to 4 decimal places. **(3)**

(d) Show that the root found in part (c) is $x = 0.441$ correct to 3 decimal places. **(2)**

(Total for Question 9 is 9 marks)

10 The curve $y = 1/x$ is defined for $1 \leq x \leq 3$. The table gives values of y.

x : 1 1.5 2 2.5 3

y : 1 ? ? 0.4 0.3333

(a) Find the missing values of y at $x = 1.5$ and $x = 2$, each to 4 decimal places. **(2)**

(b) Use the trapezium rule with all 4 strips to find an estimate for the integral of $1/x$ from 1 to 3, giving your answer to 4 decimal places. **(3)**

(c) Given that the exact value of the integral of $1/x$ from 1 to 3 is $\ln(3)$, calculate the percentage error in the estimate found in part (b), giving your answer to 2 decimal places. **(3)**

(d) By considering the second derivative of $y = 1/x$, explain why the trapezium rule overestimates the integral on this interval. **(2)**

(Total for Question 10 is 10 marks)

11 Let $f(x) = \ln(x) + x - 4$, for $x > 0$.

(a) Show that $f(x) = 0$ has a root between $x = 2$ and $x = 3$. **(2)**

(b) Show that $f'(x) = 1/x + 1$, and hence write down the Newton-Raphson iteration formula for solving $f(x) = 0$. **(2)**

(c) Using $x_0 = 3$, find x_1 and x_2 using the Newton-Raphson method, giving each answer to 6 decimal places. **(3)**

(d) Prove that the root of $f(x) = 0$ is $x = 2.93$ correct to 2 decimal places. **(2)**

(Total for Question 11 is 9 marks)

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12 For the iteration $x_{n+1} = g(x_n)$ converging to a root α, state the condition on $g'(x)$ near α.

(Total for Question 12 is 1 mark)

13 Let $f(x) = x^3 - 6x + 2$. Evaluate $f(3)$.

(Total for Question 13 is 1 mark)

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14 Show that the equation $x^3 - 6x + 2 = 0$ has a root between $x = 2$ and $x = 3$.

(Total for Question 14 is 2 marks)

15 Let $f(x) = x^3 - 3x + 1$.

(a) Show that $f'(x) = 0$ when $x = 1$, and explain why the Newton-Raphson formula cannot be used with $x_0 = 1$ to estimate a root of $f(x) = 0$. **(2)**

(b) Show that $f(x) = 0$ has a root between $x = 1.4$ and $x = 1.6$. **(2)**

(c) Using $x_0 = 1.5$, apply the Newton-Raphson method to find this root correct to 3 decimal places, showing your iterations. **(4)**

(d) Prove that this root is correct to 3 decimal places by considering a suitable sign change. **(2)**

(Total for Question 15 is 10 marks)

16 The equation $x^2 + 8x - 10 = 0$ has a positive root, α. The iteration $x_{n+1} = (10 - x_n^2)/8$ is used to estimate α, starting from $x_0 = 1$.

(a) Use the quadratic formula to show that the exact positive root of $x^2 + 8x - 10 = 0$ is $\alpha = -4 + \sqrt{26}$. **(3)**

(b) Show that $x^2 + 8x - 10 = 0$ can be rearranged to give $x = (10 - x^2)/8$. **(1)**

(c) Find $g'(x)$, and show that $|g'(\alpha)| < 1$. **(2)**

(d) By finding x_1, x_2, x_3 and x_4 (each to 4dp), show that the iteration converges to α by oscillating alternately above and below the true value (a cobweb pattern), rather than approaching α from one side only (a staircase pattern). **(3)**

(e) State how the resulting diagram would differ if g'(α) were positive instead of negative, with |g'(α)| still less than 1. **(1)**

(Total for Question 16 is 10 marks)

17 Let $f(x) = x^3 - x - 3$.

(a) Show that $f(x) = 0$ has a root between $x = 1$ and $x = 2$. **(2)**

(b) Using the iteration $x_{n+1} = (x_n + 3)^{1/3}$ with $x_0 = 1.5$, find x_1, x_2, x_3 and x_4, giving each to 4 decimal places, and state the root correct to 4 decimal places. **(4)**

(c) Show that the alternative rearrangement $x_{n+1} = x_n^3 - 3$, using the same $x_0 = 1.5$, does not converge to the root found in part (b) by finding x_1 and x_2. **(3)**

(d) Use the Newton-Raphson method with $x_0 = 1.7$ to find the root of $f(x) = 0$ correct to 4 decimal places, showing at least two iterations. **(4)**

(e) By comparing your answers to parts (b) and (d), state which method reached the root to 4 decimal places using fewer iterations. **(1)**

(Total for Question 17 is 14 marks)

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18 Using $x_0 = 1$ and the iteration $x_{n+1} = (2x_n + 5)^{1/2}$, find x_1 and x_2, each to 4 decimal places.

(Total for Question 18 is 2 marks)

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19 Let $f(x) = x^3 - 6x + 2$. Find $f'(x)$, and hence find $f'(2)$.

(Total for Question 19 is 2 marks)

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20 Let $f(x) = x^3 - 6x + 2$, with $f'(x) = 3x^2 - 6$. Using $x_0 = 2$, apply the Newton-Raphson method to find x_1, giving your answer to 4 decimal places.

(Total for Question 20 is 2 marks)

21 The table shows values of $y = 1/(x+1)$ for $x = 0, 1, 2, 3, 4$. x: 0, 1, 2, 3, 4 y: 1, 0.5, 0.3333, 0.25, 0.2. Use the trapezium rule with all four strips (h=1) to find an estimate for the integral of $1/(x+1)$ from x=0 to x=4, giving your answer to 4 decimal places.

(Total for Question 21 is 2 marks)

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22 Show that the equation $x^3 + 3x - 7 = 0$ has a root between $x = 1$ and $x = 2$. Show that this equation can be rearranged into the iterative form $x_{n+1} = (7 - 3x_n)^{1/3}$.

(Total for Question 22 is 3 marks)

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23 Using $x_0 = 1.5$, apply the iteration $x_{n+1} = (7 - 3x_n)^{1/3}$ to find x_1, x_2 and x_3, giving each value to 4 decimal places. Hence write down an estimate for the root of $x^3 + 3x - 7 = 0$, correct to 1 decimal place.

(Total for Question 23 is 3 marks)

24 Using $x_0 = 2$ and the Newton-Raphson iteration $x_{n+1} = x_n - (x_n^2 - 7)/(2x_n)$, find x_1, giving your exact answer.

(Total for Question 24 is 1 mark)

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25 Using $x_0 = 1$ and the iteration $x_{n+1} = (2x_n + 5)^{1/2}$, find x_1, giving your answer to 4 decimal places.

(Total for Question 25 is 1 mark)

26 Let $f(x) = 2\sin(x) - x + 1$, where x is measured in radians.

(a) Show that $f(x) = 0$ has a root between $x = 2.3$ and $x = 2.4$. **(2)**

(b) Find $f'(x)$, and hence write down the Newton-Raphson iteration formula for solving $f(x) = 0$. **(2)**

(c) Using $x_0 = 2.3$, find x_1 and x_2, giving each answer to 4 decimal places, and state the root of $f(x) = 0$ correct to 4 decimal places. **(4)**

(d) Show that this root is $x = 2.38$ correct to 2 decimal places. **(2)**

(Total for Question 26 is 10 marks)

27 The curve $y = \ln(x+1)$ is defined for $0 \leq x \leq 4$. The table gives values of y at x = 0, 1, 2, 3, 4. x: 0, 1, 2, 3, 4 y: 0, 0.6931, p, 1.3863, q. Find the values of p and q, each correct to 4 decimal places. Use the trapezium rule with all four strips (h=1) to find an estimate for the integral of $\ln(x+1)$ from x=0 to x=4, giving your answer to 4 decimal places.

(Total for Question 27 is 4 marks)

28 Let $f(x) = xe^x - 5$. This question compares two numerical methods for solving $f(x) = 0$.

(a) Show that $f(x) = 0$ has a root between $x = 1.3$ and $x = 1.4$. **(2)**

(b) Show that the equation $xe^x = 5$ can be rearranged into the iterative form $x_{n+1} = \ln(5/x_n)$. **(2)**

(c) Using $x_0 = 1.3$, find x_1, x_2, x_3 and x_4, giving each to 4 decimal places, and comment on the pattern of convergence. **(4)**

(d) By first finding $f'(x)$ using the product rule, use the Newton-Raphson method with $x_0 = 1.3$ to find x_1 and x_2, giving each to 6 decimal places, and state the root of $f(x) = 0$ correct to 4 decimal places. **(5)**

(e) Comment on the relative rates of convergence of the two methods used in parts (c) and (d), referring to the number of iterations required to obtain the root correct to 4 decimal places. **(2)**

(Total for Question 28 is 15 marks)

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29 The equation $x^3 - 2x - 5 = 0$ has a root α near $x = 2$. Two rearrangements are suggested: $x_{n+1} = (2x_n + 5)^{1/3}$ and $x_{n+1} = (x_n^3 - 5)/2$. Using $x_0 = 2$ for each, find x_1 and x_2 for both iterations, giving each value to 4 decimal places, and hence state, with a reason, which rearrangement converges to α.

(Total for Question 29 is 4 marks)

Mark scheme · 1.9 Pure: Numerical Methods

Question 1

(a) h = 4/5 = 0.8 **B1**

(a) trapezium rule: (h/2)[y_0 + y_n + 2(sum of the middle values)] **M1**

(a) sum of middle values = 6.3720 **A1**

(a) 6.3154 **A1**

(a) Answer: 6.3154

(b) underestimate **B1**

(b) the curve is concave (concave down), so the chords lie below the curve **B1**

(b) Answer: underestimate

(c) |estimate - exact|/exact x 100 **M1**

(c) 0.16% **A1**

(c) Answer: 0.16%

Question 2

(a) evaluates f(2) and f(3) **M1**

(a) f(2) = -7 and f(3) = 9, a change of sign, and f is continuous, so a root lies in the interval (cso) **A1**

(a) Answer: f(2) = -7 < 0 < f(3) = 9

(b) adds 3x + 9 to both sides: $x^3 = 3x + 9$ **M1**

(b) takes cube roots: $x = (3x + 9)^{1/3}$ (cso) **A1**

(b) Answer: $x = (3x + 9)^{1/3}$

(c) $x_1 = (3 \times 2 + 9)^{1/3}$ **M1**

(c) $x_1 = 2.4662$ **A1**

(c) $x_2 = 2.5406$ and $x_3 = 2.5521$ **A1**

(c) Answer: $x_1 = 2.4662$, $x_2 = 2.5406$, $x_3 = 2.5521$

(d) evaluates f(2.545) and f(2.555) **M1**

(d) f(2.545) = -0.1510 and f(2.555) = 0.0141, a change of sign, so α = 2.55 to 2 d.p. (cso) **A1**

(d) Answer: α = 2.55

Question 3

(a) h = 2/5 = 0.4 **B1**

(a) trapezium rule: (h/2)[y_0 + y_n + 2(sum of the middle values)] **M1**

(a) sum of middle values = 6.9088 **A1**

(a) 3.4464 **A1**

(a) Answer: 3.4464

(b) underestimate **B1**

(b) the curve is concave (concave down), so the chords lie below the curve **B1**

(b) Answer: underestimate

(c) |estimate - exact|/exact x 100 **M1**

(c) 0.038% (accept 0.039% using the unrounded estimate and exact value) **A1**

(c) Answer: 0.038% (accept 0.039% using the unrounded estimate and exact value)

Question 4

(a) h = 3/4 = 0.75 **B1**

(a) trapezium rule: (h/2)[y_0 + y_n + 2(sum of the middle values)] **M1**

(a) sum of middle values = 4.4841 **A1**

(a) 4.4470 **A1**

(a) Answer: 4.4470

(b) underestimate **B1**

(b) the curve is concave (concave down), so the chords lie below the curve **B1**

(b) Answer: underestimate

(c) |estimate - exact|/exact x 100 **M1**

(c) 0.17% **A1**

(c) Answer: 0.17%

Question 5

(a) differentiates each term **M1**

(a) $f'(x) = 3x^2 + 1$ **A1**

(a) Answer: $f'(x) = 3x^2 + 1$

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **M1**

(b) $x_1 = 2.1786$ **A1**

(b) $x_2 = 1.8164$ **A1**

(b) $x_3 = 1.7421$ **A1**

(b) Answer: $x_1 = 2.1786$, $x_2 = 1.8164$, $x_3 = 1.7421$

(c) α = 1.739 **B1**

(c) evaluates f(1.7385) and f(1.7395) **M1**

(c) values -0.0071 and 0.0030 have opposite signs **A1**

(c) Answer: α = 1.739

Question 6

(a) differentiates each term **M1**

(a) $f'(x) = 3x^2 + 2$ **A1**

(a) Answer: $f'(x) = 3x^2 + 2$

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **M1**

(b) $x_1 = 1.5000$ **A1**

(b) $x_2 = 1.3429$ **A1**

(b) $x_3 = 1.3284$ **A1**

(b) Answer: $x_1 = 1.5000$, $x_2 = 1.3429$, $x_3 = 1.3284$

(c) α = 1.328 **B1**

(c) evaluates f(1.3275) and f(1.3285) **M1**

(c) values -0.0056 and 0.0017 have opposite signs **A1**

(c) Answer: α = 1.328

Question 7

(a) differentiates each term **M1**

(a) $f'(x) = 1/x + 1$ **A1**

(a) Answer: $f'(x) = 1/x + 1$

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **M1**

(b) $x_1 = 0.6760$ **A1**

(b) $x_2 = 0.9646$ **A1**

(b) $x_3 = 0.9997$ **A1**

(b) Answer: $x_1 = 0.6760$, $x_2 = 0.9646$, $x_3 = 0.9997$

(c) $\alpha = 1.000$ **B1**

(c) evaluates f(0.9995) and f(1.0005) **M1**

(c) values -0.0010 and 0.0010 have opposite signs **A1**

(c) Answer: $\alpha = 1.000$

Question 8

(a) f must be continuous on [a,b] **B1**

(a) Answer: f(x) must be continuous on the interval [a,b].

(b) $x_{n+1} = x_n - f(x_n)/f'(x_n)$ **B1**

(b) Answer: $x_{n+1} = x_n - f(x_n)/f'(x_n)$

(c) f(x) = 1/(x-2) is discontinuous (has a vertical asymptote) at x = 2, which lies inside the interval [1,3] **B1**

(c) the change of sign is caused by the discontinuity, not by a root; f(x) = 0 has no solution at all here, so the method's condition (continuity) is not met, and its conclusion is invalid cso **B1**

(c) Answer: f is discontinuous at x=2 in the interval, so the sign change is due to the asymptote, not a root; the method requires continuity.

(d) near a stationary point f'(x) is close to zero, so the tangent to the curve is nearly horizontal **B1**

(d) the tangent then meets the x-axis a long way from x_0 (or, if $f'(x_0)=0$ exactly, the formula requires division by zero and cannot be used), so the next iterate can be far from the required root **B1**

(d) Answer: $f'(x_0)$ is close to (or equal to) zero, so the near-horizontal tangent gives a next estimate far from the root, or the formula is undefined.

(e) $|g'(\alpha)| < 1$ **B1**

(e) reason: this ensures the size of the error reduces at each step, so successive iterates get closer to α **B1**

(e) Answer: $|g'(\alpha)| < 1$, so that the error decreases at each iteration.

Question 9

(a) evaluates f(0) and f(1) **M1**

(a) f(0) = 3, f(1) = -3; opposite signs and f continuous, so by the change of sign rule a root lies in (0,1) cso **A1**

(a) Answer: f(0) = 3, f(1) = -3, sign change confirms a root in (0,1)

(b) states $f'(x) = 3x^2 - 7$ and substitutes $x_0 = 0.5$ into $x_1 = x_0 - f(x_0)/f'(x_0)$, using f(0.5) = -0.375 and f'(0.5) = -6.25 **M1**

(b) $x_1 = 0.5 - (-0.375/-6.25) = 0.44$ cso **A1**

(b) Answer: $x_1 = 0.44$

(c) correct Newton-Raphson substitution using $x_1 = 0.44$ **M1**

(c) $x_2 = 0.440808$ (awrt 6dp) **A1**

(c) root = 0.4408 (4dp) cao ($x_3 = 0.440808$, confirming stability) **A1**

(c) Answer: $x_2 = 0.440808$, root = 0.4408 (4dp)

(d) evaluates f(0.4405) and f(0.4415) **M1**

(d) f(0.4405) = +0.0020 (>0) and f(0.4415) = -0.0044 (<0); change of sign confirms root = 0.441 to 3dp cso **A1**

(d) Answer: root = 0.441 (3dp)

Question 10

(a) y(1.5) = 0.6667 (awrt 4dp) **B1**

(a) y(2) = 0.5 **B1**

(a) Answer: y(1.5) = 0.6667, y(2) = 0.5

(b) correct trapezium structure with h = 0.5: (h/2) $[y_0+y_4+2(y_1+y_2+y_3)]$ **M1**

(b) correct value inside the bracket, 4.4667 **A1**

(b) 1.1167 (awrt 4dp) **A1**

(b) Answer: Estimate = 1.1167

(c) attempts percentage error = |estimate - exact|/exact x 100 **M1**

(c) uses ln(3) = 1.0986 (awrt 4dp) **A1**

(c) 1.64% (awrt 2dp) **A1**

(c) Answer: Percentage error = 1.64%

(d) $d2y/dx2 = 2/x^3 > 0$ for $1 \le x \le 3$, so the curve is convex on this interval **B1**

(d) hence the chords lie above the curve, so the trapezium rule overestimates the true area **B1**

(d) Answer: y = 1/x is convex on [1,3] (d2y/dx2 > 0), so chords lie above the curve, giving an overestimate.

Question 11

(a) evaluates f(2) and f(3) **M1**

(a) f(2) = -1.3069 (<0), f(3) = 0.0986 (>0); opposite signs and f continuous on [2,3], so a root lies in (2,3) cso **A1**

(a) Answer: f(2) = -1.3069, f(3) = 0.0986, sign change confirms a root in (2,3)

(b) f'(x) = 1/x + 1, obtained by differentiating ln(x) + x - 4 term by term cso **B1**

(b) $x_{n+1} = x_n - (\ln(x_n)+x_n-4)/(1/x_n + 1)$ **B1**

(b) Answer: f'(x) = 1/x + 1; $x_{n+1} = x_n - (\ln(x_n)+x_n-4)/(1/x_n+1)$

(c) correct Newton-Raphson substitution with $x_0 = 3$ **M1**

(c) $x_1 = 2.926041$ (awrt 6dp) **A1**

(c) $x_2 = 2.926271$ (awrt 6dp) **A1**

(c) Answer: $x_1 = 2.926041$, $x_2 = 2.926271$

(d) evaluates f(2.925) and f(2.935) **M1**

(d) f(2.925) = -0.0017 (<0) and f(2.935) = +0.0117 (>0); change of sign confirms root = 2.93 to 2dp cso **A1**

(d) Answer: root = 2.93 (2dp)

Question 12

$|g'(\alpha)| < 1$ **B1**

Answer: $|g'(\alpha)| < 1$

Question 13

11 cao **B1**

Answer: 11

Question 14

evaluates f(2) = -2 and f(3) = 11 **M1**

states f(2) < 0 and f(3) > 0, a change of sign, so a root lies between x=2 and x=3 **A1**

Answer: f(2) = -2 and f(3) = 11, a sign change, so a root lies between x=2 and x=3.

Question 15

(a) $f'(x) = 3x^2 - 3$, so f'(1) = 3-3 = 0 cso **M1**

(a) the Newton-Raphson formula requires dividing by $f'(x_0)$; since f'(1) = 0 this division is undefined, so x_1 cannot be found (geometrically, the tangent at x=1 is horizontal and never meets the x-axis) **A1**

(a) Answer: f'(1) = 0, so the formula requires division by zero and cannot be used.

(b) evaluates f(1.4) and f(1.6) **M1**

(b) f(1.4) = -0.456 (<0), f(1.6) = 0.296 (>0); sign change confirms root in (1.4,1.6) cso **A1**

(b) Answer: f(1.4) = -0.456, f(1.6) = 0.296, sign change confirms a root in (1.4,1.6)

(c) correct Newton-Raphson substitution with $x_0 = 1.5$, using $f'(x) = 3x^2-3$ **M1**

(c) $x_1 = 1.5333$ (awrt 4dp) **A1**

(c) $x_2 = 1.5321$ (awrt 4dp) **A1**

(c) root = 1.532 (3dp) cao **A1**

(c) Answer: $x_1 = 1.533333$, $x_2 = 1.532091$, root = 1.532 (3dp)

(d) evaluates f(1.5315) and f(1.5325) **M1**

(d) f(1.5315) = -0.0024 (<0), f(1.5325) = +0.0017 (>0); sign change confirms root = 1.532 to 3dp cso **A1**

(d) Answer: root = 1.532 (3dp)

Question 16

(a) correct application of the quadratic formula with a=1, b=8, c=-10 **M1**

(a) $x = (-8 \pm \sqrt{64+40})/2 = -4 \pm \sqrt{26}$ **A1**

(a) takes the positive root, $\alpha = -4 + \sqrt{26}$ cso **A1**

(a) Answer: $\alpha = -4 + \sqrt{26}$

(b) correct rearrangement: $8x = 10 - x^2$, so $x = (10-x^2)/8$ cso **B1**

(b) Answer: $x = (10-x^2)/8$

(c) g'(x) = -2x/8 = -x/4 **M1**

(c) $g'(\alpha) = -1.0990/4 = -0.2748$, so $|g'(\alpha)| = 0.2748 < 1$ cso **A1**

(c) Answer: g'(x) = -x/4; $|g'(\alpha)| = 0.2748 < 1$

(d) correct repeated substitution into the iteration formula **M1**

(d) $x_1 = 1.1250$, $x_2 = 1.0918$, $x_3 = 1.1010$, $x_4 = 1.0985$ (awrt 4dp) **A1**

(d) values alternate above (1.1250, 1.1010) and below (1.0918, 1.0985) $\alpha = 1.0990$, consistent with $g'(\alpha) < 0$, giving a cobweb (not staircase) diagram cso **A1**

(d) Answer: x_1=1.1250, x_2=1.0918, x_3=1.1010, x_4=1.0985; values oscillate either side of α, confirming a cobweb pattern.

(e) the iteration would converge monotonically from one side only (a staircase diagram) rather than oscillating **B1**

(e) Answer: It would give a staircase diagram, converging from one side only.

Question 17

(a) evaluates f(1) and f(2) **M1**

(a) f(1) = -3, f(2) = 3; opposite signs and f continuous on [1,2], so a root lies in (1,2) cso **A1**

(a) Answer: f(1) = -3, f(2) = 3, sign change confirms a root in (1,2)

(b) correct repeated substitution into the iteration formula **M1**

(b) $x_1 = 1.6510$ (awrt 4dp) **A1**

(b) $x_2 = 1.6692$, $x_3 = 1.6714$ (awrt 4dp each) **A1**

(b) $x_4 = 1.6717$ and hence root = 1.6717 (4dp) cao **A1**

(b) Answer: x_1=1.6510, x_2=1.6692, x_3=1.6714, x_4=1.6717; root = 1.6717 (4dp)

(c) correct substitution $x_1 = 1.5^3 - 3$ **M1**

(c) $x_1 = 0.375$, $x_2 = 0.375^3 - 3 = -2.9473$ (awrt 4dp) **A1**

(c) the values are moving rapidly away from 1.6717, so this rearrangement diverges rather than converging cso **A1**

(c) Answer: $x_1 = 0.375$, $x_2 = -2.9473$; values diverge away from the root.

(d) states $f'(x) = 3x^2 - 1$ **M1**

(d) correct Newton-Raphson substitution with $x_0 = 1.7$ **M1**

(d) $x_1 = 1.6722$ (awrt 4dp) **A1**

(d) $x_2 = 1.6717$ and root = 1.6717 (4dp) cao **A1**

(d) Answer: $x_1 = 1.672229$, $x_2 = 1.671700$, root = 1.6717 (4dp)

(e) the Newton-Raphson method (part (d)), which reached 1.6717 after 2 iterations, compared with 4 iterations needed for the rearrangement in part (b) **B1**

(e) Answer: Newton-Raphson converged faster (2 iterations versus 4).

Question 18

finds $x_1 = 2.6458$ (or uses the unrounded value) **M1**

x_2 = awrt 3.2080 **A1**

Answer: $x_1 = 2.6458$, $x_2 = 3.2080$

Question 19

differentiates to find $f'(x) = 3x^2 - 6$ **M1**

6 cao **A1**

Answer: $f'(x) = 3x^2 - 6$; $f'(2) = 6$

Question 20

substitutes into $x_1 = x_0 - f(x_0)/f'(x_0) = 2 - (-2)/6$ **M1**

awrt 2.3333 **A1**

Answer: $x_1 = 2.3333$

Question 21

substitutes into the trapezium rule, $(1/2)[(y_0+y_4) + 2(y_1+y_2+y_3)]$ **M1**

awrt 1.6833 **A1**

Answer: 1.6833

Question 22

evaluates f(1) = -3 and f(2) = 7, a change of sign, so a root lies between x=1 and x=2 **M1**

rearranges $x^3 + 3x - 7 = 0$ to $x^3 = 7 - 3x$ **M1**

takes the cube root of both sides to give $x = (7-3x)^{1/3}$ (shown) **A1**

Answer: Root between x=1 and x=2 (shown); rearrangement $x_{n+1} = (7-3x_n)^{1/3}$ (shown)

Question 23

finds x_1 = awrt 1.3572 **M1**

finds x_2 = awrt 1.4307 and x_3 = awrt 1.3938 **A1**

1.4 cao (ft from x_3, or from the known trend of the iteration) **A1**

Answer: $x_1 = 1.3572$, $x_2 = 1.4307$, $x_3 = 1.3938$; root is approximately 1.4

Question 24

2.75 cao, oe 11/4 **B1**

Answer: $x_1 = 2.75$

Question 25

awrt 2.6458 **B1**

Answer: $x_1 = 2.6458$

Question 26

(a) evaluates f(2.3) and f(2.4) in radians **M1**

(a) f(2.3) = +0.1914 (>0), f(2.4) = -0.0491 (<0); sign change confirms root in (2.3,2.4) cso **A1**

(a) Answer: f(2.3) = 0.1914, f(2.4) = -0.0491, sign change confirms a root in (2.3,2.4)

(b) $f'(x) = 2\cos(x) - 1$ **B1**

(b) $x_{n+1} = x_n - (2\sin(x_n)-x_n+1)/(2\cos(x_n)-1)$ **B1**

(b) Answer: $f'(x) = 2\cos(x) - 1$; $x_{n+1} = x_n - (2\sin(x_n)-x_n+1)/(2\cos(x_n)-1)$

(c) correct Newton-Raphson substitution with $x_0 = 2.3$ **M1**

(c) x_1 = 2.3821 (awrt 4dp) **A1**

(c) x_2 = 2.3801 (awrt 4dp) **A1**

(c) root = 2.3801 (4dp) cao **A1**

(c) Answer: $x_1 = 2.3821$, $x_2 = 2.3801$, root = 2.3801 (4dp)

(d) evaluates f(2.375) and f(2.385) **M1**

(d) f(2.375) = +0.0124 (>0), f(2.385) = -0.0121 (<0); sign change confirms root = 2.38 to 2dp cso **A1**

(d) Answer: root = 2.38 (2dp)

Question 27

p = ln(3) = awrt 1.0986 **B1**

q = ln(5) = awrt 1.6094 **B1**

substitutes into the trapezium rule, $(1/2)[(y_0+y_4) + 2(y_1+y_2+y_3)]$ **M1**

awrt 3.9828 **A1**

Answer: p = 1.0986, q = 1.6094; trapezium estimate = 3.9828

Question 28

(a) evaluates f(1.3) and f(1.4) **M1**

(a) f(1.3) = -0.2299 (<0), f(1.4) = +0.6773 (>0); sign change confirms a root in (1.3,1.4) cso **A1**

(a) Answer: f(1.3) = -0.2299, f(1.4) = 0.6773, sign change confirms a root in (1.3,1.4)

(b) rearranges $xe^x = 5$ to $e^x = 5/x$ **M1**

(b) takes ln of both sides to give $x = \ln(5/x)$ cso **A1**

(b) Answer: $x_{n+1} = \ln(5/x_n)$

(c) correct repeated substitution into the iteration formula **M1**

(c) $x_1 = 1.3471$ (awrt 4dp) **A1**

(c) $x_2 = 1.3115$, $x_3 = 1.3383$ (awrt 4dp each) **A1**

(c) $x_4 = 1.3181$ (awrt 4dp), with a valid comment that the values oscillate above and below the root and are converging only slowly **A1**

(c) Answer: x_1=1.3471, x_2=1.3115, x_3=1.3383, x_4=1.3181; the iteration oscillates and converges slowly.

(d) $f'(x) = e^x + xe^x = e^x(1+x)$ (product rule) **B1**

(d) correct Newton-Raphson substitution with $x_0 = 1.3$ **M1**

(d) $x_1 = 1.327243$ (awrt 6dp) **A1**

(d) $x_2 = 1.326725$ (awrt 6dp) **A1**

(d) root = 1.3267 (4dp) cao **A1**

(d) Answer: $f'(x)=e^x(1+x)$; x_1=1.327243, x_2=1.326725, root = 1.3267 (4dp)

(e) the Newton-Raphson method converges much faster, giving the root to 4dp after only 2 iterations, whereas the iteration in (c) is still oscillating and has not settled to 4dp after 4 iterations **B1**

(e) valid supporting reason, e.g. Newton-Raphson uses gradient information to make a much closer next estimate at each step, so fewer iterations are needed for the same accuracy **B1**

(e) Answer: Newton-Raphson converges far more quickly (2 iterations vs. more than 4) because it uses the gradient of f to target the root directly.

Question 29

finds x_1 = awrt 2.0801 and x_2 = awrt 2.0924 for $x_{n+1} = (2x_n+5)^{1/3}$ **M1**

finds $x_1 = 1.5000$ and $x_2 = -0.8125$ for $x_{n+1} = (x_n^3-5)/2$ **A1**

compares the two sequences: the first is settling towards a value near 2.09, the second is moving further away from x_0 **M1**

concludes $x_{n+1} = (2x_n+5)^{1/3}$ converges to α (the values are approaching a fixed value), while $x_{n+1} = (x_n^3-5)/2$ diverges (the values move away from the root) **A1**

Answer: First iteration converges to α (approximately 2.09); second iteration diverges