2.7 Algebraic Proof: 2-hour Pack

AQA 8365 · Calculator allowed · about 120 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 n is an integer. Prove that $(2n + 5)^2 - 25$ is always a multiple of 8.

(Total for Question 1 is 4 marks)

2 Show that $(n + 7)^2 - (n + 3)^2$ is always a multiple of 4.

(Total for Question 2 is 4 marks)

3 n is an integer. Prove that the sum of 7 consecutive integers n, n + 1, n + 2, ... is always a multiple of 7.

(Total for Question 3 is 4 marks)

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4 n is a positive integer. Prove that $n(n + 4) + 4$ is always a square number.

(Total for Question 4 is 4 marks)

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5 n is an integer. Prove that $(n + 12)(n + 9)$ is always an even number.

(Total for Question 5 is 3 marks)

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6 Show that $(n + 6)(n - 6) - n(n)$ is a constant, and find its value.

(Total for Question 6 is 4 marks)

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7 n is a positive integer. Prove that $n(n + 16) + 64$ is always a square number.

(Total for Question 7 is 4 marks)

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8 12n and 12(n + 1) are two consecutive multiples of 12, where n is an integer. Prove that the difference between their squares is always an odd multiple of 144.

(Total for Question 8 is 4 marks)

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9 n is an integer. Prove that (n + 9)(n + 6) is always an even number.

(Total for Question 9 is 3 marks)

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10 Show that (n + 9)(n + 4) - n(n + 13) is a constant, and find its value.

(Total for Question 10 is 4 marks)

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11 n is an integer. Prove that $(2n + 9)^2 - 81$ is always a multiple of 8.

(Total for Question 11 is 4 marks)

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12 n is an integer. Prove that the sum of 9 consecutive integers n, n + 1, n + 2, ... is always a multiple of 9.

(Total for Question 12 is 4 marks)

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13 n is an integer. Prove that (n + 2)(n + 1) is always an even number.

(Total for Question 13 is 3 marks)

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14 8n and 8(n + 1) are two consecutive multiples of 8, where n is an integer. Prove that the difference between their squares is always an odd multiple of 64.

(Total for Question 14 is 4 marks)

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15 8n and 8(n + 1) are two consecutive multiples of 8, where n is an integer. Prove that the sum of their squares is a multiple of 64.

(Total for Question 15 is 4 marks)

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16 6n and 6(n + 1) are two consecutive multiples of 6, where n is an integer. Prove that the difference between their squares is always an odd multiple of 36.

(Total for Question 16 is 4 marks)

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17 Two consecutive odd numbers can be written as 2n - 1 and 2n + 1, where n is an integer. Prove that the difference between the squares of two consecutive odd numbers is always a multiple of 8.

(Total for Question 17 is 3 marks)

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18 Let x = 0.454545... (recurring, where the block '45' repeats forever). Prove that x = 5/11.

(Total for Question 18 is 3 marks)

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19 Let n be a positive integer. Prove that $n^3 + n$ is always even.

(Total for Question 19 is 2 marks)

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20 Prove that $x^2 - 8x + 20 > 0$ for all real values of x.

(Total for Question 20 is 3 marks)

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21 Prove that the equation $x^2 - 4x + 8 = 0$ has no real roots.

(Total for Question 21 is 3 marks)

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22 Two numbers differ by 4 and can be written as $2n + 3$ and $2n - 1$, where n is an integer. Prove that the difference between their squares is always a multiple of 8.

(Total for Question 22 is 2 marks)

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23 The equation $ax^2 + bx + c = 0$ has $a > 0$, and $b^2 < 4ac$. Prove that the equation has no real roots.

(Total for Question 23 is 4 marks)

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24 Let n be an integer. Prove that the sum of four consecutive integers is always even but never a multiple of 4.

(Total for Question 24 is 4 marks)

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25 Prove by contradiction that $\sqrt{2}$ is irrational.

(Total for Question 25 is 5 marks)

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26 Every positive integer n can be written as n = 10k + r, where k is an integer and r is the units digit of n (so r is one of 0, 1, 2, 3, 4, 5, 6, 7, 8, 9). Prove that a square number can only end in 0, 1, 4, 5, 6 or 9, and can never end in 2, 3, 7 or 8.

(Total for Question 26 is 6 marks)

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27 Prove by contradiction that there is no largest even integer.

(Total for Question 27 is 4 marks)

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28 Prove by contradiction that $\sqrt{3}$ is irrational.

(Total for Question 28 is 4 marks)

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Mark scheme · 2.7 Algebraic Proof: 2-hour Pack

Question 1

expands: $4n^2 + 20n + 25$	**M1**
subtracts 25: $4n^2 + 20n$	**M1**
$4n(n + 5)$	**A1**
n and n + 5 differ by an odd number so one is even; hence 4 x even is a multiple of 8	**C1**

Answer: $4n(n + 5)$

Question 2

expands both brackets: $n^2 + 14n + 49$ and $n^2 + 6n + 9$	**M1**
n^2 terms cancel: $8n + 40$	**M1**
$4(2n + 10)$ or equivalent factorised form	**A1**
conclusion linking the factorised form to the claim	**C1**

Answer: $4(2n + 10)$

Question 3

sum = n + (n + 1) + ... + (n + 6) written down	**M1**
collects to $7n + 21$	**M1**
$7(n + 3)$	**A1**
conclusion: 7 is a factor, so the sum is a multiple of 7	**C1**

Answer: $7(n + 3)$

Question 4

expands: $n^2 + 4n + 4$	**M1**
recognises $2^2 = 4$ and 2 x 2 = 4	**M1**
$(n + 2)^2$	**A1**
$(n + 2)^2$ is the square of the integer n + 2	**C1**

Answer: $(n + 2)^2$

Question 5

n even: (n + 12) is even, so the product is even	**M1**
n odd: (n + 9) is even, so the product is even	**M1**
both cases covered, so the product is always even	**A1**

Answer: always even

Question 6

expands the first product (3 terms correct)	**M1**
expands n(...) and subtracts	**M1**
n^2 and n terms cancel; result -36	**A1**
states the value is independent of n	**C1**

Answer: -36

Question 7

expands: $n^2 + 16n + 64$	**M1**
recognises $8^2 = 64$ and 2 x 8 = 16	**M1**
$(n + 8)^2$	**A1**
$(n + 8)^2$ is the square of the integer n + 8	**C1**

Answer: $(n + 8)^2$

Question 8

$(12(n + 1))^2 - (12n)^2$	**M1**
$144(n^2 + 2n + 1) - 144n^2$ or equivalent	**M1**
$12^2(2n + 1)$	**A1**
2n + 1 is odd, so the result is an odd multiple of 144	**C1**

Answer: $12^2(2n + 1)$

Question 9

n even: (n + 6) is even, so the product is even	**M1**
n odd: (n + 9) is even, so the product is even	**M1**
both cases covered, so the product is always even	**A1**

Answer: always even

Question 10

expands the first product (3 terms correct)	**M1**
expands n(...) and subtracts	**M1**
n^2 and n terms cancel; result 36	**A1**
states the value is independent of n	**C1**

Answer: 36

Question 11

expands: $4n^2 + 36n + 81$	**M1**
subtracts 81: $4n^2 + 36n$	**M1**
$4n(n + 9)$	**A1**
n and n + 9 differ by an odd number so one is even; hence 4 x even is a multiple of 8	**C1**

Answer: $4n(n + 9)$

Question 12

sum = n + (n + 1) + ... + (n + 8) written down	**M1**
collects to $9n + 36$	**M1**
$9(n + 4)$	**A1**
conclusion: 9 is a factor, so the sum is a multiple of 9	**C1**

Answer: $9(n + 4)$

Question 13

n even: (n + 2) is even, so the product is even	**M1**
n odd: (n + 1) is even, so the product is even	**M1**
both cases covered, so the product is always even	**A1**

Answer: always even

Question 14

$(8(n + 1))^2 - (8n)^2$	**M1**
$64(n^2 + 2n + 1) - 64n^2$ or equivalent	**M1**
$8^2(2n + 1)$	**A1**
2n + 1 is odd, so the result is an odd multiple of 64	**C1**

Answer: $8^2(2n + 1)$

Question 15

$(8n)^2 + (8(n + 1))^2$	**M1**
$64n^2 + 64(n^2 + 2n + 1)$ or equivalent	**M1**
$8^2(2n^2 + 2n + 1)$	**A1**
64 is a factor of every term, so the sum is a multiple of 64	**C1**

Answer: $8^2(2n^2 + 2n + 1)$

Question 16

$(6(n + 1))^2 - (6n)^2$	**M1**
$36(n^2 + 2n + 1) - 36n^2$ or equivalent	**M1**
$6^2(2n + 1)$	**A1**
2n + 1 is odd, so the result is an odd multiple of 36	**C1**

Answer: $6^2(2n + 1)$

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Question 17

uses difference of two squares: $(2n+1)^2 - (2n-1)^2 = [(2n+1)-(2n-1)][(2n+1)+(2n-1)]$	**M1**
simplifies correctly to $(2)(4n) = 8n$	**A1**
correct conclusion that 8n is always a multiple of 8 for integer n	**A1**

Answer: $(2n+1)^2 - (2n-1)^2 = 8n$, a multiple of 8

Question 18

multiplies both sides by 100 (as two digits repeat), giving $100x = 45.454545...$	**M1**
subtracts $x = 0.454545...$ from $100x$ to eliminate the recurring part, giving $99x = 45$	**M1**
solves and simplifies fully to $x = 45/99 = 5/11$	**A1**

Answer: x = 5/11

Question 19

factorises n^3+n as $n(n^2+1)$	**M1**
cso: valid reasoning that if n is even the product is even, and if n is odd then n^2 is odd so n^2+1 is even, making the product even; so n^3+n is always even	**A1**

Answer: $n^3 + n = n(n^2+1)$, always even

Question 20

completes the square correctly, $x^2 - 8x + 20 = (x-4)^2 - 16 + 20$	**M1**
simplifies to $(x-4)^2 + 4$	**A1**
cso: since $(x-4)^2 \geq 0$ for all real x, $(x-4)^2 + 4 \geq 4 > 0$, so $x^2-8x+20 > 0$ for all real x	**A1**

Answer: $x^2 - 8x + 20 = (x-4)^2 + 4 > 0$ for all real x

Question 21

discriminant $= (-4)^2 - 4(1)(8)$ (= 16-32)	**M1**
= -16	**A1**
cso: since $-16 < 0$ the equation has no real roots	**A1**

Answer: Discriminant = -16 < 0, so no real roots

Question 22

uses difference of two squares: $(2n+3)^2 - (2n-1)^2 = [(2n+3)-(2n-1)][(2n+3)+(2n-1)]$	**M1**
cso: simplifies correctly to $(4)(4n+2) = 16n+8 = 8(2n+1)$, a multiple of 8	**A1**

Answer: $(2n+3)^2 - (2n-1)^2 = 8(2n+1)$, a multiple of 8

Question 23

completes the square correctly, $a(x + b/(2a))^2 + c - b^2/(4a) = 0$	**M1**
rearranges to isolate the squared term, $a(x + b/(2a))^2 = (b^2 - 4ac)/(4a)$	**A1**
valid reasoning that the left-hand side is always ≥ 0 (since $a > 0$), while the right-hand side is negative when $b^2 < 4ac$ (as $4a > 0$)	**A1**
correct conclusion: a non-negative quantity cannot equal a negative quantity, so no real value of x satisfies the equation, i.e. no real roots exist	**A1**

Answer: No real roots exist when $b^2 < 4ac$ and $a > 0$

Question 24

writes the sum of four consecutive integers as $n+(n+1)+(n+2)+(n+3)$, simplifying to $4n + 6$	**M1**
factorises correctly to $2(2n+3)$	**A1**
valid reasoning that $2n + 3$ is always odd (2n is even, plus the odd number 3)	**A1**
correct conclusion that 2 x (an odd number) is always even, but is never divisible by 4, since $(2n+3)$ is never even	**A1**

Answer: Sum = 2(2n+3), always even but never a multiple of 4

Question 25

assumes, for contradiction, that $\sqrt{2}$ is rational, so $\sqrt{2} = p/q$ where p and q are integers with no common factor (in lowest terms), q not equal to 0	**B1**
squares both sides to get $p^2 = 2q^2$, and deduces that since p^2 is even, p itself must be even	**B1**
substitutes $p = 2k$ into $p^2 = 2q^2$ to obtain $4k^2 = 2q^2$, so $q^2 = 2k^2$, and deduces q must also be even	**B1**
states that p and q being both even contradicts the assumption that p/q was in lowest terms (no common factor)	**B1**
correct final conclusion: the contradiction means the original assumption was false, so $\sqrt{2}$ is irrational	**B1**

Answer: $\sqrt{2}$ is irrational (proof by contradiction)

Question 26

Level 1 Attempts to expand n^2 = (10k+r)^2, or lists some values of r^2, but with limited progress and no valid general conclusion.	**1-2**
Level 2 Expands (10k+r)^2 correctly and shows the units digit of n^2 depends only on r^2. Checks several but not all values of r from 0 to 9, or makes a minor error in the case check, but reaches a broadly correct partial conclusion.	**3-4**
Level 3 Fully correct and complete argument: expands (10k+r)^2 to show the units digit of n^2 depends only on the units digit of r^2, systematically checks all ten values r = 0 to 9, correctly lists the resulting units digits (0, 1, 4, 9, 6, 5, 6, 9, 4, 1), and concludes clearly that 2, 3, 7 and 8 never appear, so a square number can never end in these digits.	**5-6**

Indicative content:

- n^2 = (10k+r)^2 = 100k^2 + 20kr + r^2 = 10(10k^2 + 2kr) + r^2
- The units digit of n^2 is therefore determined entirely by the units digit of r^2 (since 10(10k^2+2kr) contributes only to tens and higher place values)
- Systematically evaluating r^2 for r = 0,1,2,...,9 gives: 0, 1, 4, 9, 16, 25, 36, 49, 64, 81
- Taking units digits of these values gives: 0, 1, 4, 9, 6, 5, 6, 9, 4, 1
- None of these units digits is 2, 3, 7 or 8
- Since every integer n has a units digit r in {0,...,9}, and all cases have been checked, no square number can ever end in 2, 3, 7 or 8

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Question 27

assumes, for contradiction, that a largest even integer exists, and calls it E	**B1**
considers E + 2, and states that E + 2 is even, since E is even and 2 is even, so their sum is even	**B1**
shows that E + 2 > E, since adding a positive number always gives a larger number	**B1**
cso: correct conclusion: E + 2 is an even integer larger than E, contradicting the assumption that E was the largest even integer, so no largest even integer exists	**B1**

Answer: No largest even integer exists (proof by contradiction)

Question 28

assumes, for contradiction, that $\sqrt{3}$ is rational, so $\sqrt{3} = p/q$ where p and q are integers with no common factor (in lowest terms), q not equal to 0; squares both sides to get $p^2 = 3q^2$, and deduces that since p^2 is divisible by 3, p itself must be divisible by 3	**B1**
substitutes $p = 3k$ into $p^2 = 3q^2$ to obtain $9k^2 = 3q^2$, so $q^2 = 3k^2$, and deduces q must also be divisible by 3	**B1**
states that p and q being both divisible by 3 contradicts the assumption that p/q was in lowest terms (no common factor)	**B1**
cso: correct final conclusion: the contradiction means the original assumption was false, so $\sqrt{3}$ is irrational	**B1**

Answer: $\sqrt{3}$ is irrational (proof by contradiction)

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