2.7 Algebraic Proof: Test Pack

AQA 8365 · Calculator allowed · about 60 minutes

Total Marks

Name: _____ Date: ____ / ____ / _____

Answer ALL questions. Show all your working.

1 Let n be an integer. Which of the following always represents an even number?

- ☐ A) 2n + 1
- ☐ B) 2n
- ☐ C) n + 1
- ☐ D) 2n - 1

(Total for Question 1 is 1 mark)

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2 Let n be an integer. Show that the sum of two consecutive integers is always odd.

(Total for Question 2 is 2 marks)

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3 Let n be an integer.

(a) Show that $n^2 + (n+1)^2 = 2n^2 + 2n + 1$. **(1)**

(b) Hence prove that the sum of the squares of two consecutive integers is always odd. **(1)**

(Total for Question 3 is 2 marks)

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4 Let n be an integer.

(a) Show that $(n + 2)^2 - n^2 = 4n + 4$. **(2)**

(b) Hence prove that the difference between the squares of any two integers that differ by 2 is always a multiple of 4. **(2)**

(Total for Question 4 is 4 marks)

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5 Prove that $x^2 + 4x + 7 > 0$ for all real values of x.

(Total for Question 5 is 3 marks)

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6 Aisha attempts to prove that $n^2 + n$ is always even for any integer n. Her working is shown below:

"$n^2 + n = n(n+1)$. Since n+1 is even, n(n+1) must be even."

(Total for Question 6 is 2 marks)

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7 11n and 11(n + 1) are two consecutive multiples of 11, where n is an integer. Prove that the difference between their squares is always an odd multiple of 121.

(Total for Question 7 is 4 marks)

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8 n is an integer. Prove that $(2n + 1)^2 - 1$ is always a multiple of 8.

(Total for Question 8 is 4 marks)

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9 n is an integer. Prove that $(n + 8)(n + 5)$ is always an even number.

(Total for Question 9 is 3 marks)

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10 Show that $(n - 6)(n + 8) - n(n + 2)$ is a constant, and find its value.

(Total for Question 10 is 4 marks)

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11 9n and 9(n + 1) are two consecutive multiples of 9, where n is an integer. Prove that the sum of their squares is a multiple of 81.

(Total for Question 11 is 4 marks)

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12 n is a positive integer. Prove that $n(n + 18) + 81$ is always a square number.

(Total for Question 12 is 4 marks)

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13 Let $x = 0.363636...$ (recurring, where the block '36' repeats forever). Prove that $x = 4/11$.

(Total for Question 13 is 2 marks)

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14 Two consecutive odd numbers can be written as $2n + 1$ and $2n + 3$, where n is an integer. Prove that the sum of the squares of two consecutive odd numbers is always even but never a multiple of 4.

(Total for Question 14 is 4 marks)

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15 Show that $(2n)^2 + (n^2 - 1)^2 = (n^2 + 1)^2$ for all positive integers n.

(Total for Question 15 is 3 marks)

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16 A student claims: "$n^2 - n + 11$ is a prime number for every positive integer n." Prove or disprove the claim.

(Total for Question 16 is 3 marks)

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17 Prove by contradiction that there is no smallest positive rational number.

(Total for Question 17 is 4 marks)

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Mark scheme · 2.7 Algebraic Proof: Test Pack

Question 1

B cao **B1**

Answer: B

Question 2

writes the sum of two consecutive integers as n + (n+1), simplifying to 2n + 1 oe **M1**

correct conclusion that 2n is always even, so 2n + 1 is always odd, hence the sum of two consecutive integers is always odd **A1**

Answer: n + (n+1) = 2n + 1, which is odd for every integer n

Question 3

(a) cso: expands $(n+1)^2$ correctly to n^2+2n+1 and adds n^2 to give $2n^2+2n+1$, answer printed **B1**

(a) Answer: $2n^2 + 2n + 1$

(b) cso: correct conclusion that $2n^2+2n = 2(n^2+n)$ is always even, so $2n^2+2n+1$ is always odd **B1**

(b) Answer: $2n^2 + 2n + 1 = 2(n^2+n) + 1$, always odd

Question 4

(a) expands $(n+2)^2$ correctly to $n^2 + 4n + 4$ **M1**

(a) subtracts n^2 to give 4n + 4 (oe 4(n+1)), answer printed **A1**

(a) Answer: 4n + 4

(b) factorises their result from (a) as 4(n+1) **M1**

(b) correct conclusion that since n+1 is an integer, 4(n+1) is always a multiple of 4 **A1**

(b) Answer: 4n + 4 = 4(n+1), a multiple of 4

Question 5

completes the square correctly, $x^2 + 4x + 7 = (x+2)^2 - 4 + 7$ **M1**

simplifies to $(x+2)^2 + 3$ **A1**

valid conclusion: since $(x+2)^2 \geq 0$ for all real x, $(x+2)^2 + 3 \geq 3 > 0$, so $x^2+4x+7 > 0$ always **A1**

Answer: $x^2 + 4x + 7 = (x+2)^2 + 3 > 0$ for all real x

Question 6

identifies the error: n+1 is not always even (e.g. if n is even, such as n=4, then n+1=5 is odd), so the claim 'n+1 is even' is a false general statement **B1**

gives a correct justification: n and n+1 are consecutive integers, so exactly one of them must always be even, meaning their product n(n+1) is always even regardless of which one is even **B1**

Answer: The claim that n+1 is always even is false; the correct reasoning is that one of n, n+1 (consecutive integers) is always even

Question 7

$(11(n + 1))^2 - (11n)^2$ **M1**

$121(n^2 + 2n + 1) - 121n^2$ or equivalent **M1**

$11^2(2n + 1)$ **A1**

2n + 1 is odd, so the result is an odd multiple of 121 **C1**

Answer: $11^2(2n + 1)$

Question 8

expands: $4n^2 + 4n + 1$ **M1**

subtracts 1: $4n^2 + 4n$ **M1**

4n(n + 1) **A1**

n and n + 1 differ by an odd number so one is even; hence 4 x even is a multiple of 8 **C1**

Answer: 4n(n + 1)

Question 9

n even: (n + 8) is even, so the product is even **M1**

n odd: (n + 5) is even, so the product is even **M1**

both cases covered, so the product is always even **A1**

Answer: always even

Question 10

expands the first product (3 terms correct) **M1**

expands n(...) and subtracts **M1**

n^2 and n terms cancel; result -48 **A1**

states the value is independent of n **C1**

Answer: -48

Question 11

$(9n)^2 + (9(n + 1))^2$ **M1**

$81n^2 + 81(n^2 + 2n + 1)$ or equivalent **M1**

$9^2(2n^2 + 2n + 1)$ **A1**

81 is a factor of every term, so the sum is a multiple of 81 **C1**

Answer: $9^2(2n^2 + 2n + 1)$

Question 12

expands: $n^2 + 18n + 81$ **M1**

recognises $9^2 = 81$ and 2 x 9 = 18 **M1**

$(n + 9)^2$ **A1**

$(n + 9)^2$ is the square of the integer n + 9 **C1**

Answer: $(n + 9)^2$

Question 13

multiplies both sides by 100 (as two digits repeat), giving 100x = 36.3636..., and subtracts x to eliminate the recurring part, giving 99x = 36 **M1**

cso: solves and simplifies fully to x = 36/99 = 4/11 **A1**

Answer: x = 4/11

Question 14

expands both squares correctly, $(2n+1)^2 = 4n^2+4n+1$ and $(2n+3)^2 = 4n^2+12n+9$ **M1**

adds and simplifies to $8n^2 + 16n + 10$ (oe $2(4n^2+8n+5)$) **A1**

valid reasoning that $4n^2 + 8n + 5$ is always odd (since $4n^2$ and 8n are both even, plus the odd number 5) **A1**

correct conclusion that 2 x (an odd number) is always even, but can never be a multiple of 4 **A1**

Answer: $(2n+1)^2 + (2n+3)^2 = 2(4n^2+8n+5)$, always even but never a multiple of 4

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Question 15

expands $(n^2-1)^2$ correctly to $n^4 - 2n^2 + 1$	**M1**
adds $(2n)^2 = 4n^2$ to the previous expansion	**M1**
cso: simplifies the sum to $n^4 + 2n^2 + 1$, which matches the expansion of $(n^2+1)^2$, confirming the identity holds for all positive integers n	**A1**

Answer: $(2n)^2 + (n^2-1)^2 = (n^2+1)^2$ for all positive integers n

Question 16

selects a suitable value of n to test, e.g. n = 11	**M1**
evaluates n^2-n+11 correctly for the chosen n, writing it in a factorisable form, e.g. $n^2-n+11 = n(n-1)+11$, so at n=11: $11(10)+11 = 11(11)$	**M1**
cso: correct conclusion that 11 x 11 = 121 has factors other than 1 and itself, so it is not prime, disproving the claim by counter-example	**A1**

Answer: False: at n=11, n^2-n+11 = 11 x 11 = 121, which is not prime

Question 17

assumes, for contradiction, that a smallest positive rational number exists, and calls it q, where q > 0	**B1**
considers q/2, and states that q/2 is rational (since q is rational and 2 is a non-zero rational number) and q/2 > 0 (since q > 0)	**B1**
shows that q/2 < q, since dividing a positive number by 2 always gives a smaller positive number	**B1**
correct conclusion: q/2 is a positive rational number smaller than q, contradicting the assumption that q was the smallest positive rational number, so no smallest positive rational number exists	**B1**

Answer: No smallest positive rational number exists (proof by contradiction)

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